Calculus (1)

MATH 101

Course Description :

Chapter 1 : Precalculus Review .

Chapter 2 : Limits of Functions .

Chapter 3 : The Derivative .

Chapter 4 : Applications of the Derivatives .

Chapter 5 : Integrals .

Chapter 7 : Logarithmic and Exponential Functions .

Chapter 8 : Inverse Trigonometric and Hyperbolic Functions .

Text Book :

Earl W. Sowokowski , '' Calculus '' , Thomson Advantage Books, 5^{th} Edition , 1991 .

Grade Distribution :

Periodic Tests (2 Tests)	*40%*
Quiz , Attendance , Participation & Homework	*20%*
Final Exam	*40 %*

Prepared by Department of Mathematics

5/11/1434H

Chapter (1)

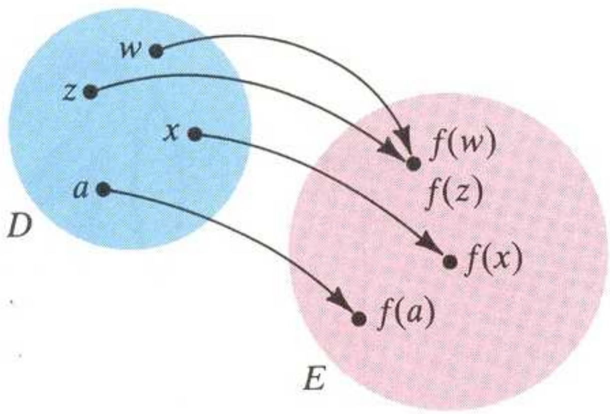
Precalculus Review

1.2 Functions :

Definition (1.10) : Page (14)

> *A function f from a set D to a set E is a correspondence that assigns to each element x of D exactly one element y of E .*

Figure 1.15



* *The element y of E is the value of f at x and is defined by $f(x)$.*

* *The set D is the domain of the function .*

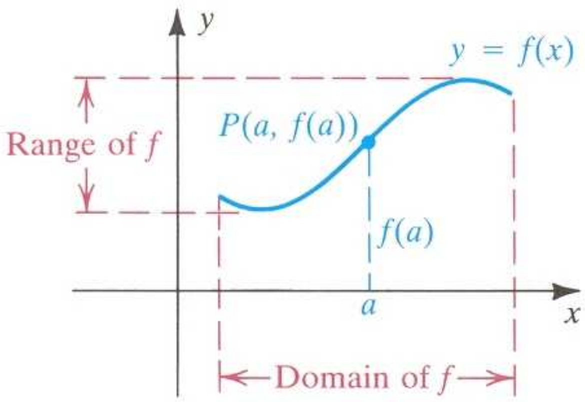
* *The range of f is the subset of E consisting of all possible function values $f(x)$ for x in D .*

* *The element x is called independent variable and y is called dependent variable .*

Notes (1) : Page (16)

* *The graph of a function f is the graph of the equation $y = f(x)$ for x in the domain of f .*

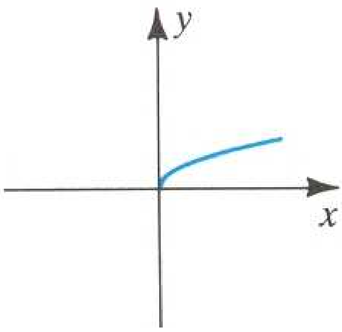
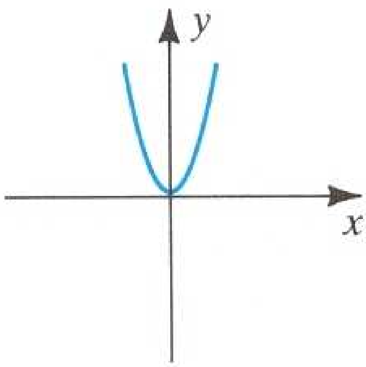
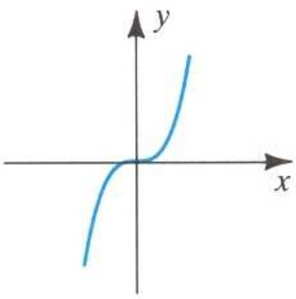
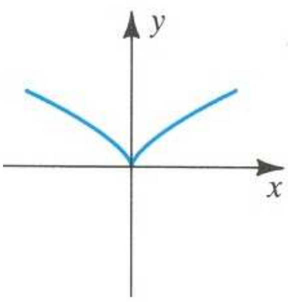
Figure 1.17

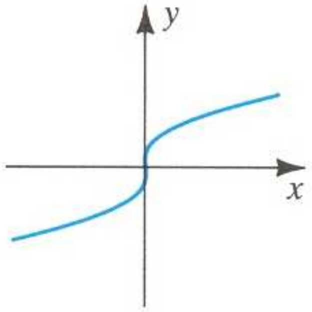
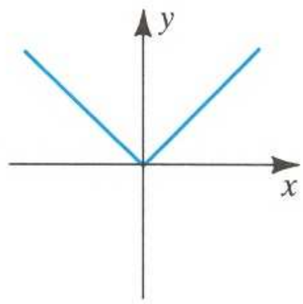
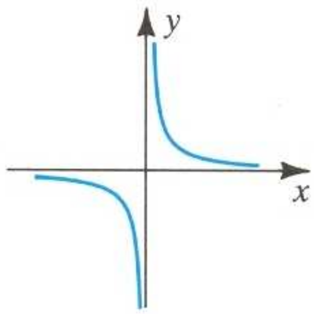


* If f is an even function – that is , if $f(-x)=f(x)$ for every x in the domain of f – then the graph of f is symmetric with respect to the y-axis .

* If f is an odd function – that is , if $f(-x)=-f(x)$ for every x in the domain of f – then the graph of f is symmetric with respect to the origin .

ILLUSTRATION : Page (16)

Function f	Graph	Symmetric	Domain D, Range R
$f(x)=\sqrt{x}$		none	$D=[0,\infty)$ $R=[0,\infty)$
$f(x)=x^2$		y-axis (even function)	$D=(-\infty,\infty)$ $R=[0,\infty)$
$f(x)=x^3$		origin (odd function)	$D=(-\infty,\infty)$ $R=(-\infty,\infty)$
$f(x)=x^{2/3}$		y-axis (even function)	$D=(-\infty,\infty)$ $R=[0,\infty)$

$f(x) = x^{1/3}$		origin (odd function)	$D = (-\infty, \infty)$ $R = (-\infty, \infty)$
$f(x) = \lvert x \rvert$		y-axis (even function)	$D = (-\infty, \infty)$ $R = [0, \infty)$
$f(x) = \dfrac{1}{x}$		origin (odd function)	$D = (-\infty, 0) \cup (0, \infty)$ $R = (-\infty, 0) \cup (0, \infty)$

Note (2) : *Page (17)*

* *Functions that are described by more than one expressions , as in the next example , are called piecewise-defined functions .*

Example (3) : *Page (17)*

Sketch the graph of the function f defined as follows :

$$f(x) = \begin{cases} 2x + 3 & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \le x < 2 \\ 1 & \text{if} \quad x \ge 2 \end{cases}$$

Solution

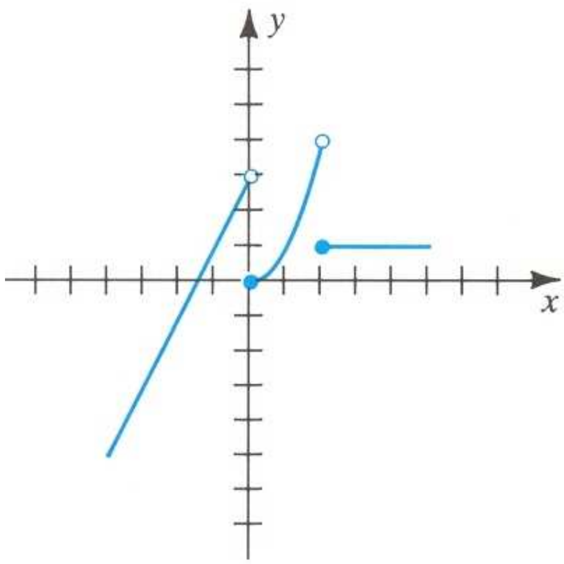
$$f(x) = \begin{cases} 2x+3 & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \le x < 2 \\ 1 & \text{if} \quad x \ge 2 \end{cases}$$

* *If* $x < 0$ *then* $f(x) = 2x+3$ *, and the graph of* f *is part of the line* $y = 2x+3$ *, as indicated in Figure 1.18 . The small circle indicates that* $(0, 3)$ *is not on the graph .*

* *If* $0 \le x < 2$ *then* $f(x) = x^2$ *, and the graph of* f *is part of the parabola* $y = x^2$. *Note that* $(2, 4)$ *is not on the graph .*

* *If* $x \ge 2$ *then* $f(x) = 1$ *, and the graph of* f *is a horizontal half-line with endpoint* $(2, 1)$.

Figure 1.18



Note (3) : *Page (18)*

* *The greatest integer function* f *is defined by* $f(x) = [\![x]\!]$.

If x *is a real number , we define the symbol* $[\![x]\!]$ *as follows :*

$$[\![x]\!] = n \text{ , where } n \text{ is the greatest integer such that } n \le x .$$

Example (4) : *Page (18)*

Sketch the graph of the greatest integer function .

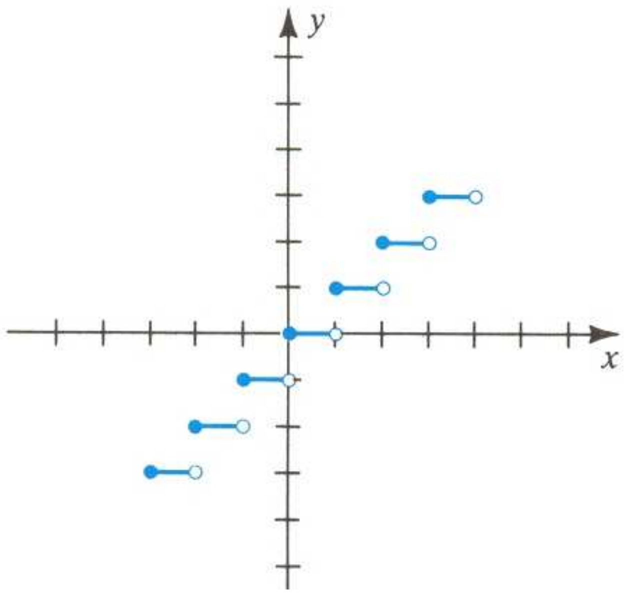
Solution

* *The x- and y-coordinates of some points on the graph may be listed as follows :*

Values of x	$f(x) = [\![x]\!]$
$\vdots$	$\vdots$
$-2 \le x < -1$	-2
$-1 \le x < 0$	-1
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2
$\vdots$	$\vdots$

***** *Part of the graph is sketched in* ***Figure 1.19*** **.**

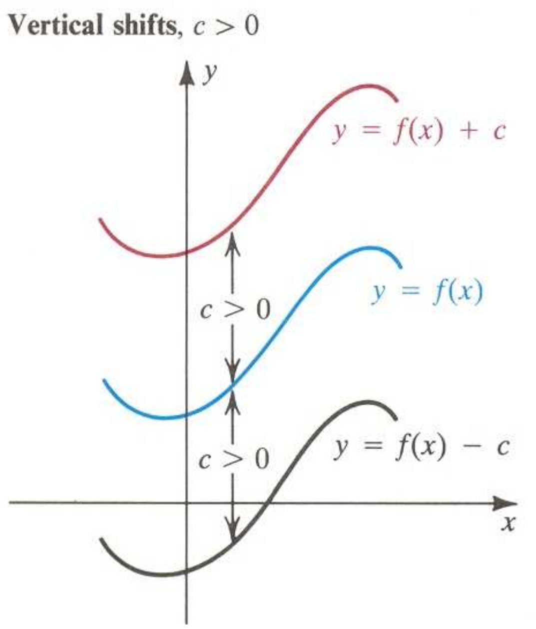
Figure 1.19



Notes (4) : ***Page (18)***

***** ***The graphs in Figure 1.20 illustrate vertical shifts of*** $y = f(x)$ ***resulting from adding a positive constant*** c ***to each function value*** $f(x)$ ***or subtracting*** c ***from*** $f(x)$ **.**

Figure 1.20



*** *Horizontal shifts* *are illustrated in* *Figure 1.21* *.***

Figure 1.21

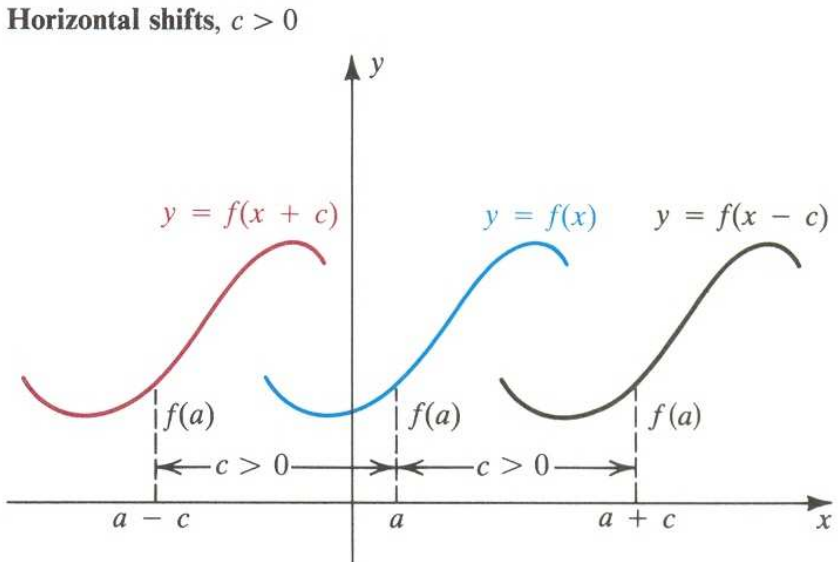


Figure 1.22

vertical shifts

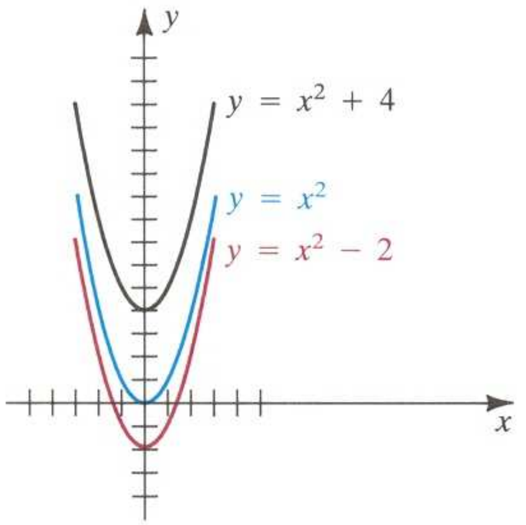


Figure 1.23

Horizontal shifts

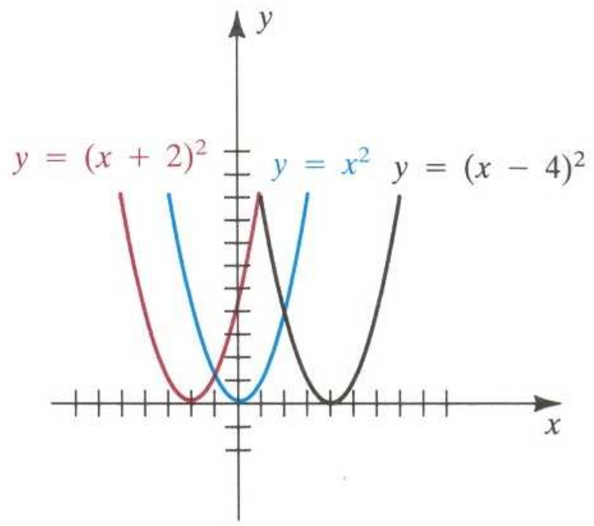


Figure 1.24

Vertical stretching and compressing

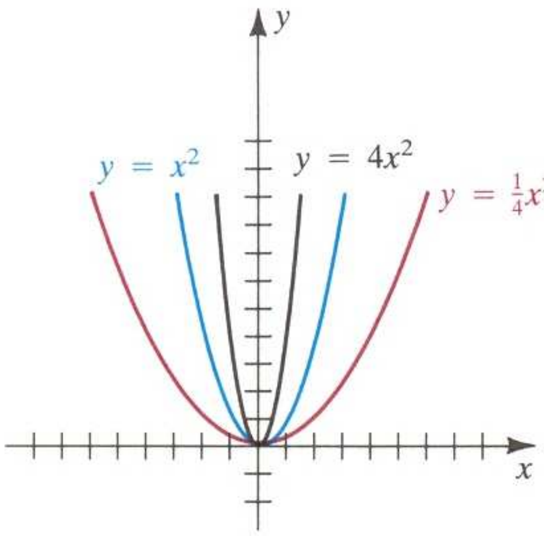
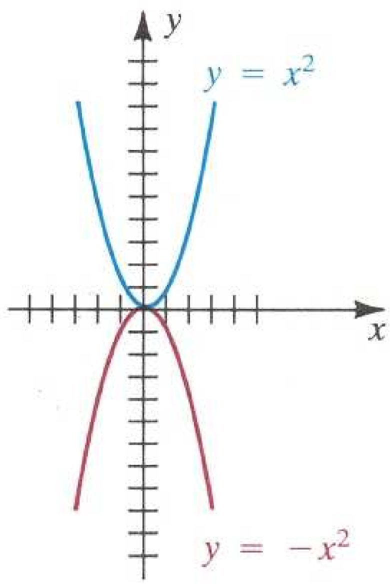


Figure 1.25

Reflection about the x-axis



Notes (5) : Page (20)

* A function f is a polynomial function if $f(x)$ is a polynomial – that is , if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where the coefficients $a_n , a_{n-1} , \dots , a_1 , a_0$ are real numbers and the exponents are nonnegative integers . If $a_n \neq 0$, then f has degree n .

$\Rightarrow$ The domain of polynomial function is $\mathbb{R}$ or $(-\infty, \infty)$.

* A rational function is a quotient of two polynomial functions – that is , if

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomial functions .

$\Rightarrow$ The domain of rational function is

$$\mathbb{R} - \{ \text{set of zeros of } Q(x) \} .$$

* A root function is in the form

$$f(x) = \sqrt[n]{g(x)}$$

where $g(x)$ is a polynomial function , n is positive integer .

$\Rightarrow$ *The domain of root function is*

$$\begin{cases} \mathbb{R} \text{ if } n \text{ is odd} \\ \text{the solution set of inequality } g(x) \geq 0 \text{ if } n \text{ is even} \end{cases}.$$

Notes (6) : ***Page (20)***

* *If f and g are functions, we define the sum $f+g$, difference $f-g$, product fg , and quotient f/g as follows :*

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)\,g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

* *The domain of $f+g$, $f-g$, and fg is the intersection of the domains of f and g - that is, the numbers that are common to both domains,*

$$\text{or} \quad D_f \cap D_g$$

* *The domain of f/g consists of all numbers x in the intersection such that $g(x) \neq 0$,*

$$\text{or} \quad D_f \cap D_g - \{\text{set of zeros of } g(x)\}$$

Example (5) : ***Page (20)***

Let $f(x) = \sqrt{4-x^2}$ and $g(x) = 3x+1$. Find the sum, difference, product, and quotient of f and g, and specify the domain of each.

Solution

* $f(x) = \sqrt{4-x^2}$ $\Rightarrow$ *Root function, to get the domain of f*

put $4-x^2 \geq 0 \Rightarrow 4 \geq x^2 \Rightarrow -2 \leq x \leq 2$

$D_f = [-2, 2]$

* $g(x) = 3x + 1 \quad \Rightarrow$ *Polynomial function*

The domain of g is $D_g = \mathbb{R} = (-\infty, \infty)$

* $D_f \cap D_g = [-2, 2] \cap \mathbb{R} = [-2, 2]$ *or* $-2 \le x \le 2$

Function	*Domain*
$(f+g)(x) = \sqrt{4-x^2} + (3x+1)$	$-2 \le x \le 2$
$(f-g)(x) = \sqrt{4-x^2} - (3x+1)$	$-2 \le x \le 2$
$(f\,g)(x) = \sqrt{4-x^2}\,(3x+1)$	$-2 \le x \le 2$
$\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{4-x^2}}{3x+1}$	$-2 \le x \le 2$ *and* $x \ne -\dfrac{1}{3}$

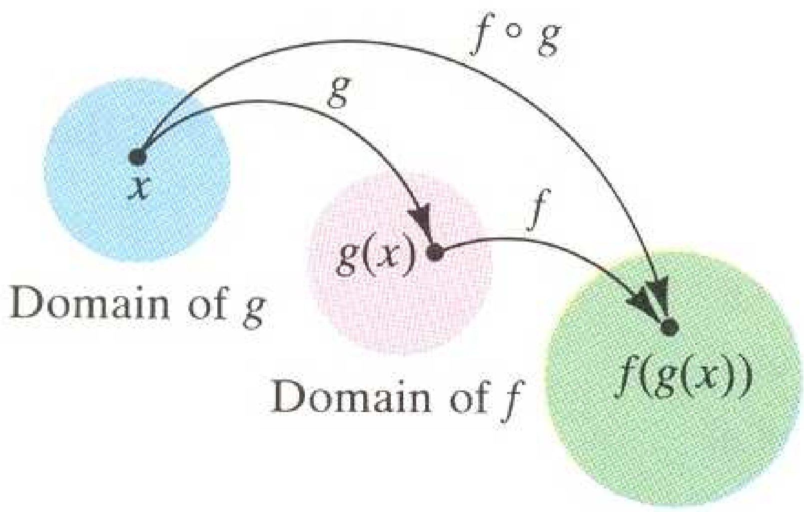
Definition (1.11) : *Page (21)*

The composite function $f \circ g$ of f and g is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Figure 1.26



Notes (7) : **Page (21)**

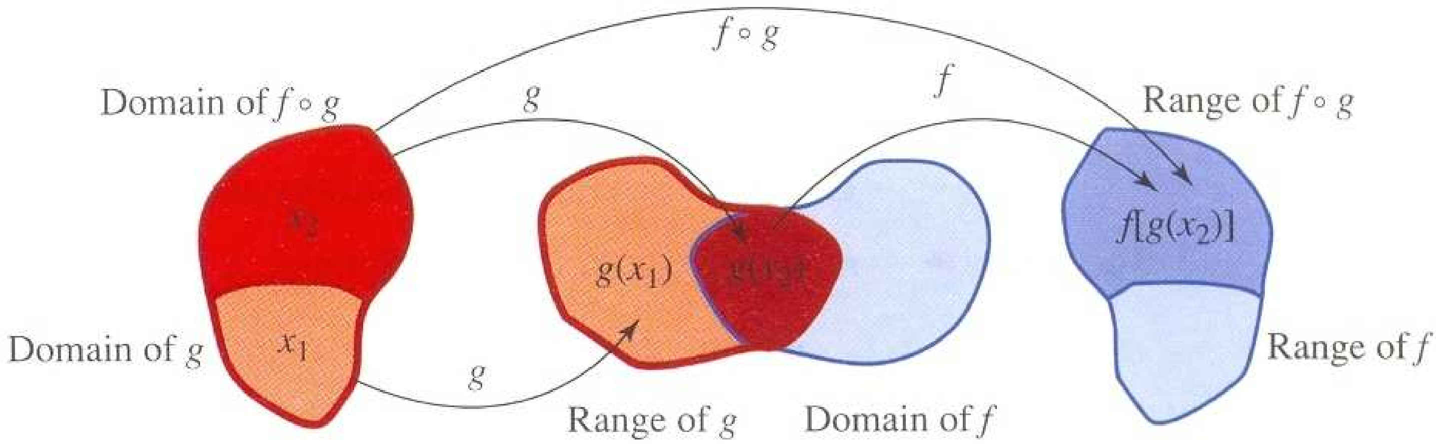
* ***The composite function*** $f \circ g$ ***(read f circle g) .***

* $(f \circ g)(x) \neq (g \circ f)(x)$.

* ***The condition to obtain*** $f \circ g$ ***is***

$$R_g \cap D_f \neq \varnothing \quad (\varnothing \text{ is null or empty set }) .$$

* $D_{f \circ g} \subseteq D_g$.



Example (6) : **Page (21)**

If $f(x) = x^2 - 1$ ***and*** $g(x) = 3x + 5$ ***, find***

(a) $(f \circ g)(x)$ ***and the domain of*** $f \circ g$.

(b) $(g \circ f)(x)$ ***and the domain of*** $g \circ f$.

Solution

* $f(x) = x^2 - 1 \quad \Rightarrow$ ***Polynomial function***

$$D_f = \mathbb{R} \quad \text{or} \quad (-\infty, \infty).$$

* $g(x) = 3x + 5 \Rightarrow$ *Polynomial function*

$$D_g = \mathbb{R} \quad \text{or} \quad (-\infty, \infty).$$

(a) $(f \circ g)(x) = f(g(x))$

$$= f(3x + 5)$$

$$= (3x + 5)^2 - 1$$

$$= \boxed{9x^2 + 30x + 24}.$$

* *Since for each x in $\mathbb{R}$ (the domain of g) the function value $g(x) = 3x + 5$ is in $\mathbb{R}$ (the domain of f), the domain of $f \circ g$ is also $\boxed{\mathbb{R}}$.*

(b) $(g \circ f)(x) = g(f(x))$

$$= g(x^2 - 1)$$

$$= 3(x^2 - 1) + 5$$

$$= \boxed{3x^2 + 2}.$$

* *Since for each x in $\mathbb{R}$ (the domain of f) the function value $f(x) = x^2 - 1$ is in $\mathbb{R}$ (the domain of g), the domain of $g \circ f$ is also $\boxed{\mathbb{R}}$.*

Example (7) : *Page (22)*

If $f(x) = x^2 - 16$ and $g(x) = \sqrt{x}$, find

(a) $(f \circ g)(x)$ and the domain of $f \circ g$.

(b) $(g \circ f)(x)$ and the domain of $g \circ f$.

Solution

* $f(x) = x^2 - 16 \Rightarrow$ *Polynomial function*

$$D_f = \mathbb{R} \quad \text{or} \quad (-\infty, \infty).$$

* $g(x)=\sqrt{x}$ $\Rightarrow$ *Root function , to get the domain of f*

put $x \geq 0$

$$D_g=[0,\infty) \quad \text{or} \quad (-\infty,\infty).$$

(a) $(f \circ g)(x)=f(g(x))$

$$=f\left(\sqrt{x}\right)$$

$$=\left(\sqrt{x}\right)^2-16$$

$$=\boxed{x+16}.$$

* *Since for each* x *in* $[0,\infty)$ *(the domain of* g *) the function value* $g(x)=\sqrt{x}$ *is in* $\mathbb{R}$ *(the domain of* f *) , the domain of*

$f \circ g$ *is* $\boxed{[0,\infty)}$.

(b) $(g \circ f)(x)=g(f(x))$

$$=g\left(x^2-16\right)$$

$$=\boxed{\sqrt{x^2-16}}.$$

* *Since for each* x *in* $\mathbb{R}$ *(the domain of* f *) the function value* $f(x)=x^2-16$ *is in* $[0,\infty)$ *(the domain of* g *) , the domain of* $g \circ f$ *is all* x *such that* x^2-16 *is in* $[0,\infty)$.

$$x^2-16 \geq 0 \quad , \quad x^2 \geq 16 \quad , \text{ and } \quad |x| \geq 4 \quad \Rightarrow \quad -4 \leq x \leq 4 .$$

Thus , the domain of $g \circ f$ *is the union* $\boxed{(-\infty,-4]\cup[4,\infty)}$.

__Notes (8) :__ *Page (22)*

* *If* f *and* g *are functions such that*

$$y=f(u) \quad \text{and} \quad u=g(x)$$

then substituting for u *in* $y=f(u)$ *yields*

$$y = f\left(g\left(x\right)\right)$$

Example (8) : Page (22)

Express $y = \left(2x+5\right)^8$ in a composite function form .

Solution

* $u = 2x+5$ and $y = u^8$

is a composite function form for $y = \left(2x+5\right)^8$

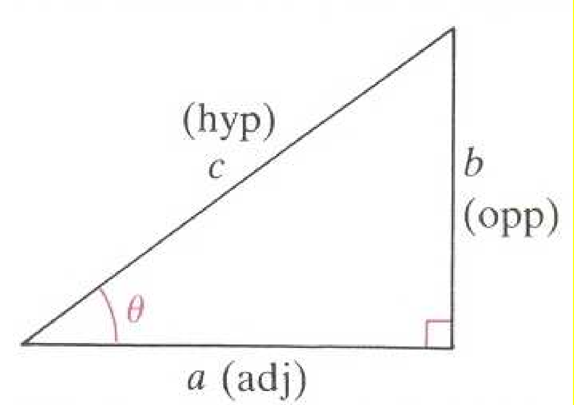
ILLUSTRATION : Page (23)

Function value	Choice for $u = g\left(x\right)$	Choice for $y = f\left(u\right)$
$y = \left(x^3 - 5x + 1\right)^4$	$u = x^3 - 5x + 1$	$y = u^4$
$y = \sqrt{x^2 - 4}$	$u = x^2 - 4$	$y = \sqrt{u}$
$y = \dfrac{2}{3x+7}$	$u = 3x+7$	$y = \dfrac{2}{u}$

1.3 TRIGONOMETRY :

The trigonometric functions (1.16) : Page (29)

(i) Of an Acute Angle θ :

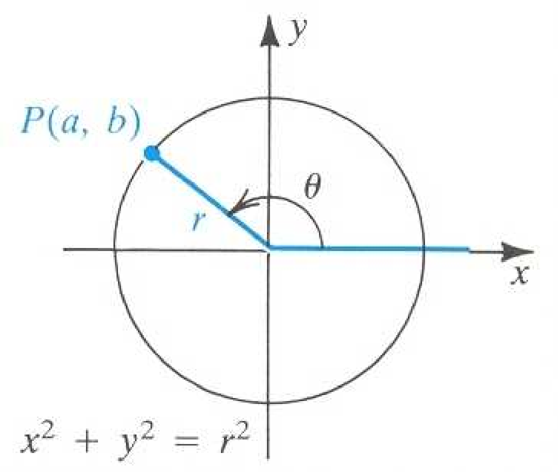


$\sin\theta = \dfrac{b}{c}$ $\csc\theta = \dfrac{c}{b}$

$\cos\theta = \dfrac{a}{c}$ $\sec\theta = \dfrac{c}{a}$

$\tan\theta = \dfrac{b}{a}$ $\cot\theta = \dfrac{a}{b}$

(ii) Of any Angle θ :



$$\sin\theta = \frac{b}{r} \qquad \csc\theta = \frac{r}{b}$$

$$\cos\theta = \frac{a}{r} \qquad \sec\theta = \frac{r}{a}$$

$$\tan\theta = \frac{b}{a} \qquad \cot\theta = \frac{a}{b}$$

(iii) Of a Real Number x :

The value of a trigonometric function at a real number x is its value at an angle of x radians .

Definition (1.17) : *Page (30)*

* $\csc\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$,

  $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, $\tan\theta = \dfrac{1}{\cot\theta}$.

* $\sin^2\theta + \cos^2\theta = 1$,

  $\tan^2\theta + 1 = \sec^2\theta$,

  $1 + \cot^2\theta = \csc^2\theta$.

Notes (9) :

* $\cos 2\theta = \cos^2\theta - \sin^2\theta$

  $= 2\cos^2\theta - 1$

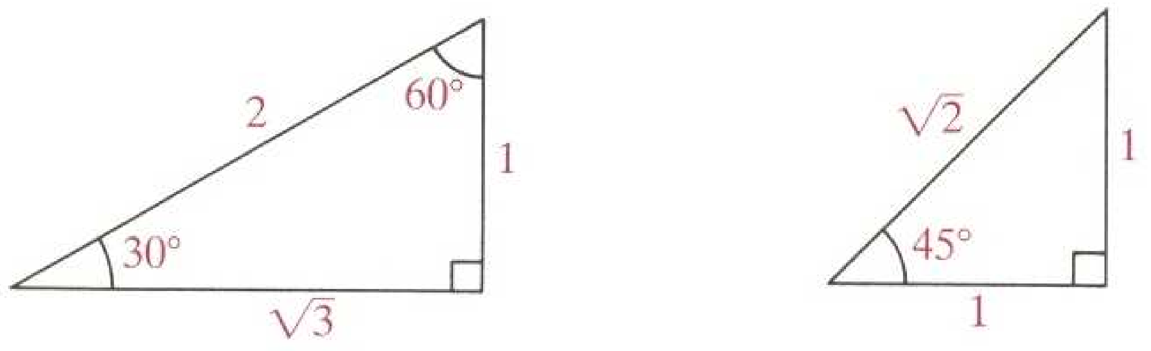
  $= 1 - 2\sin^2\theta$

* $\sin 2\theta = 2\sin\theta\cos\theta$

Special values of the trigonometric functions (1.18) : Page (31)

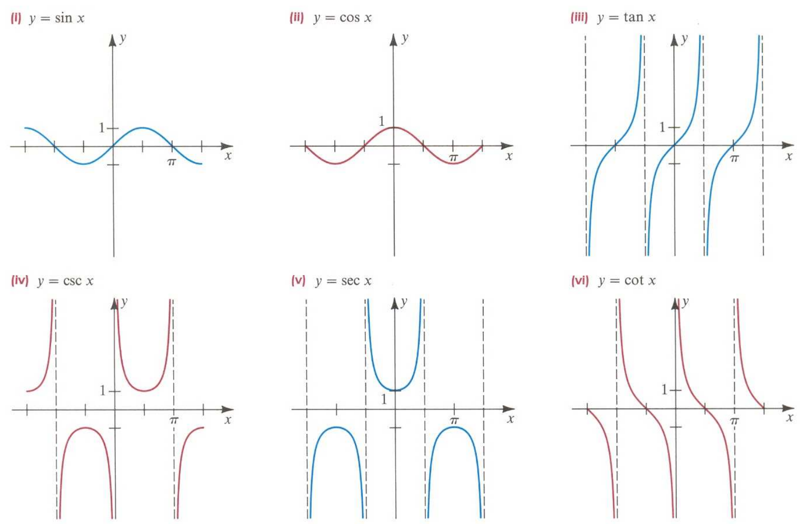
θ (Rad.)	θ (Deg.)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\csc\theta$	$\csc\theta$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

Figure 1.33



Graphs of the trigonometric functions : Page (34)

Figure 1.37



First Homework

Page 23, No. 7-10, 12, 14

Page 24, 19, 25, 35, 38, 40 and 50

CHAPTER (2)

LIMITS OF FUNCTIONS

2.1 INTRODUCTION TO LIMITS : Page (40)

* *As an illustration , consider*

$$f(x) = \frac{x^3 - 2x^2}{3x - 6}$$

* *Note that* 2 *is not in the domain of* f *, since substituting* $x = 2$ *gives us the undefined expression* $\frac{0}{0}$.

x	$f(x)$
1.9	1.20333333
1.99	1.32003333
1.999	1.33200033
1.9999	1.33320000
1.99999	1.33332000
1.999999	1.33333200

x	$f(x)$
2.1	1.47000000
2.01	1.34670000
2.001	1.33466700
2.0001	1.33346667
2.00001	1.33334667
2.000001	1.33333467

* *It appears that the closer* x *to* 2 *, the closer* $f(x)$ *to* $\frac{4}{3}$.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^3 - 2x^2}{3x - 6} = \boxed{\frac{4}{3}}.$$

* *In general*

$$f(x)=\frac{x^3-2x^2}{3x-6}$$

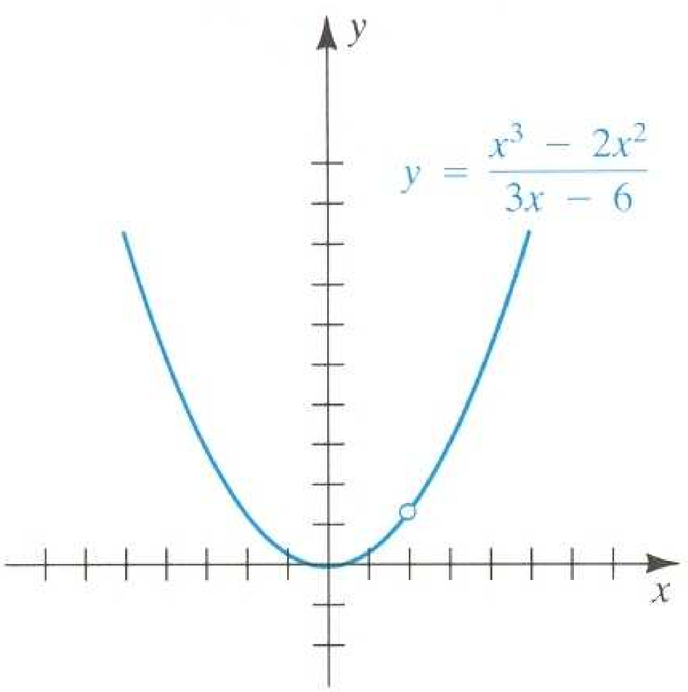
* *The number 2 is not in the domain of f since the meaningless expression $\frac{0}{0}$ is obtained if 2 is substituted for x.*
* *Factoring the numerator and denominator*

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{x^2(x-2)}{3(x-2)}$$

* *Since $x\neq 2$, we may cancel the common factor $(x-2)$*

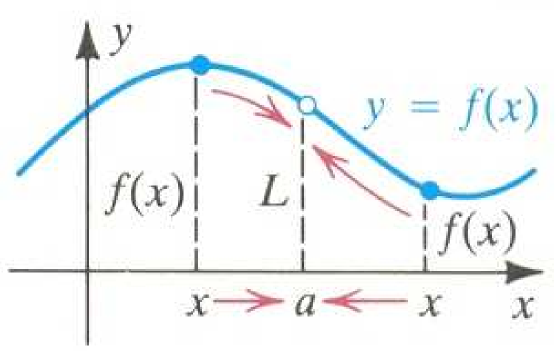
$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{x^2}{3}=\frac{2^2}{3}=\boxed{\frac{4}{3}}.$$

Figure 2.1



Limits of a function (2.1) : *Page (41)*

NOTATION	INTUITIVE MEANING	GRAPHICAL INTERPRETATION

$\lim\limits_{x \to a} f(x) = L$	*We can make* $f(x)$ *as close to* L *as desired by choosing* x *sufficiently close to* a, *and* $x \neq a$.	

Example (1) : *Page (44)*

If $f(x) = \dfrac{2x^2 - 5x + 2}{5x^2 - 7x - 6}$, *find* $\lim\limits_{x \to 2} f(x)$.

Solution

$$f(x) = \frac{2x^2 - 5x + 2}{5x^2 - 7x - 6}$$

* *The number* 2 *is not in the domain of* f *since the meaningless expression* $\dfrac{0}{0}$ *is obtained if* 2 *is substituted for* x .

* *Factoring the numerator and denominator*

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x-2)(2x-1)}{(x-2)(5x+3)}$$

* *Since* $x \neq 2$, *we may cancel the common factor* $(x-2)$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{2x-1}{5x+3} = \frac{2(2)-1}{5(2)+3} = \boxed{\frac{3}{13}} .$$

Example (2) : *Page (44)*

If $f(x) = \dfrac{x-9}{\sqrt{x} - 3}$.

(a) find $\lim\limits_{x \to 9} f(x)$.

(b) Sketch the graph of f and illustrate the limit in part (a) graphically .

Solution

(a)
$$f(x)=\frac{x-9}{\sqrt{x}-3}$$

* *The number 9 is not in the domain of f since the meaningless expression $\frac{0}{0}$ is obtained if 9 is substituted for x .*

* *Rationalizing the denominator by multiplying the numerator and denominator by $\sqrt{x}+3$*

$$\lim_{x\to 9} f(x)=\lim_{x\to 9}\left(\frac{x-9}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3}\right)$$

$$=\lim_{x\to 9}\frac{(x-9)\left(\sqrt{x}+3\right)}{x-9}$$

* *Since $x\neq 9$, we may cancel the common factor $(x-9)$*

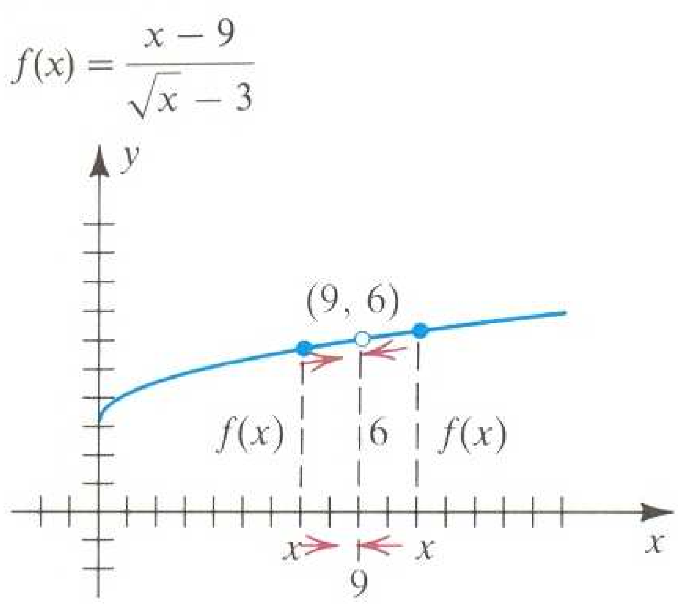
$$\lim_{x\to 9} f(x)=\lim_{x\to 9}\left(\sqrt{x}+3\right)=\sqrt{9}+3=\boxed{6}.$$

(b) The graph of f is the same as the graph of the equation $y=\sqrt{x}+3$, except for the point $(9,6)$, as illustrated in Figure 2.3 .

* *As x gets closer to 9 , the point $(x,f(x))$ on the graph of f gets closer to the point $(9,6)$.*

* *Note that $f(x)$ never actually attains the value 6 ; however , $f(x)$ can be made as close to 6 as desired by choosing x sufficiently close to 9 .*

Figure 2.3



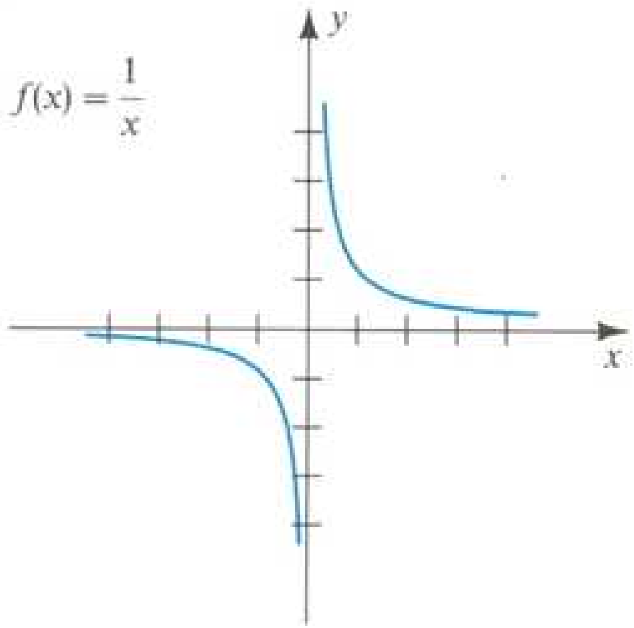
Example (3) : Page (45)

Show that $\lim\limits_{x \to 0} \frac{1}{x}$ does not exist .

Solution

* The graph of $f\,(x) = \frac{1}{x}$ is sketched in Figure 2.4 .

Figure 2.4

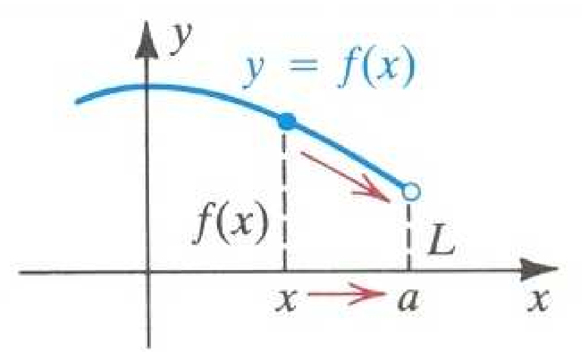
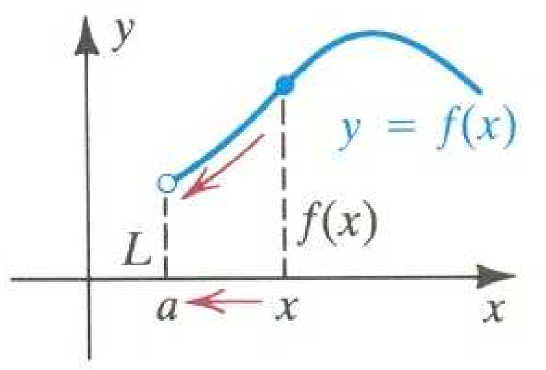


* Note that we can make $\left| f\,(x) \right|$ as large as desired by choosing x sufficiently close to 0 (but $x \neq 0$) .

* For example , if we choose $x = -0.000001$, we obtain $f(x) = 1{,}000{,}000$ and if we choose $x = 0.000001$, we obtain $f(x) = 1{,}000{,}000$.

* Since $f(x)$ does not approach a specific number as x approaches 0 , **the lim it does not exist** .

Limits of a function (2.2) : Page (46)

NOTATION	INTUITIVE MEANING	GRAPHICAL INTERPRETATION
$\lim_{x \to a^-} f(x) = L$ (left –hand limit)	We can make $f(x)$ as close to L as desired by choosing x sufficiently close to a , and $x < a$.	
$\lim_{x \to a^+} f(x) = L$ (right –hand limit)	We can make $f(x)$ as close to L as desired by choosing x sufficiently close to a , and $x > a$.	

Theorem (2.3) : Page (46)

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Example (6) : Page (47)

If $f(x) = \dfrac{|x|}{x}$, sketch the graph of f and find , if possible ,

(a) $\lim_{x \to 0^-} f(x)$. (b) $\lim_{x \to 0^+} f(x)$. (c) $\lim_{x \to 0} f(x)$.

Solution

$$f(x)=\frac{|x|}{x}$$

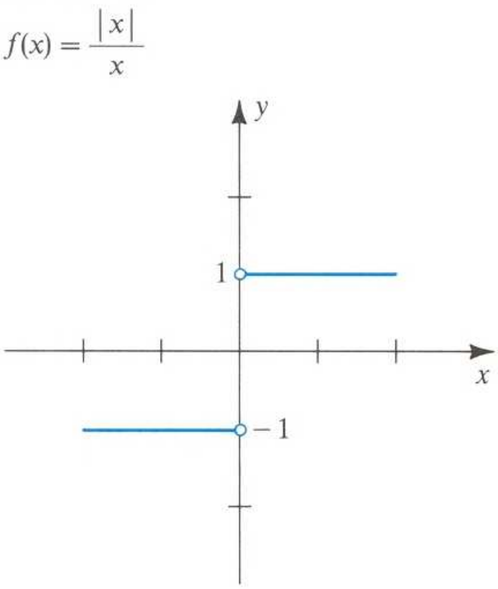
* *Since* $|x|=\begin{cases} x & \text{if } x\ge 0 \\ -x & \text{if } x<0 \end{cases}$

* f *is undefined*, $\frac{0}{0}$, *at* $x=0$.

* *If* $x>0$, *the* $|x|=x$ *and* $f(x)=\frac{|x|}{x}=\frac{x}{x}=1$.

* *If* $x<0$, *the* $|x|=-x$ *and* $f(x)=\frac{|x|}{x}=\frac{-x}{x}=-1$.

Figure 2.7



(a) $\lim\limits_{x\to 0^-} f(x)=\boxed{-1}$.

(b) $\lim\limits_{x\to 0^+} f(x)=\boxed{1}$.

(c) *Since* $\lim\limits_{x\to 0^-} f(x)\ne\lim\limits_{x\to 0^+} f(x)$, *then*

$$\boxed{\lim_{x\to 0} f(x) \text{ does not exist}}\ .$$

Example (7) : *Page (47)*

Sketch the graph of the function f defined as follows :

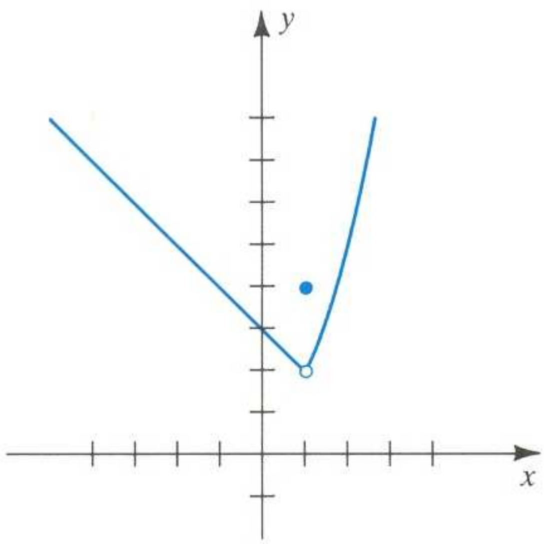
$$f(x)=\begin{cases} 3-x & \text{for} \quad x<1 \\ 4 & \text{for} \quad x=1 \\ x^2+1 & \text{for} \quad x>1 \end{cases}$$

Find $\lim\limits_{x \to 1^-} f(x)$, $\lim\limits_{x \to 1^+} f(x)$, *and* $\lim\limits_{x \to 1} f(x)$.

Solution

$$f(x)=\begin{cases} 3-x & \text{for} \quad x<1 \\ 4 & \text{for} \quad x=1 \\ x^2+1 & \text{for} \quad x>1 \end{cases}$$

Figure 2.8



* *If* $x<1$, *then* $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (3-x) = 3-1 = \boxed{2}$.

* *If* $x>1$, *then* $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x^2+1) = 1^2+1 = \boxed{2}$.

* *Since* $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = 2$, *then*

$$\boxed{\lim_{x \to 1} f(x) = 2}\ .$$

* *Note that the function value* $f(1) = 4$ *is irrelevant in finding the limit .*

2.2 DEFINITION OF LIMIT : *Page (53)*

Definition of limit of a function (2.4) : *Page (53)*

Let a function f be defined on an open interval containing a , except possibly at a itself , and let L be a real number . The statement

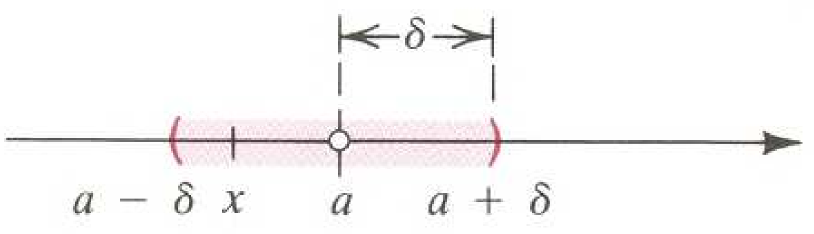
$$\lim_{x \to a} f(x) = L$$

means that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

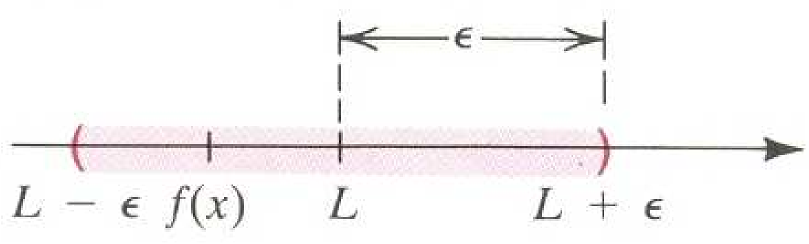
$$\text{if} \quad 0 < |x - a| < \delta \ , \quad \text{then} \quad |f(x) - L| < \varepsilon \ .$$

Figure 2.11

(i) $0 < |x - a| < \delta$



(ii) $|f(x) - L| < \varepsilon$



Example (1) : *Page (54)*

Use definition of limit of a function to prove that $\lim_{x \to 4} (3x - 5) = 7$.

Solution

* Let $f(x) = 3x - 5$, $a = 4$, and $L = 7$, the we must show that given any $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta , \quad \text{then} \quad |f(x) - L| < \varepsilon .$$

* To solve inequality problems of this type we can often obtain a clue to a proper choice for δ by examining the ε −tolerance statement as follows :

$$|(3x - 5) - 7| < \varepsilon$$

$$|3x - 12| < \varepsilon$$

$$|3(x - 4)| < \varepsilon$$

$$|(x - 4)| < \frac{1}{3}\varepsilon$$

* We choose δ such that $\delta \le \frac{1}{3}\varepsilon$ and obtain the following equivalent inequalities :

$$0 < |x - 4| < \delta$$

$$0 < |x - 4| < \frac{1}{3}\varepsilon$$

$$0 < 3|x - 4| < \varepsilon$$

$$0 < |3x - 12| < \varepsilon$$

$$0 < |(3x - 5) - 7| < \varepsilon$$

* This verifies

$$\boxed{\text{If} \quad 0 < |x - 4| < \delta , \quad \text{then} \quad |(3x - 5) - 7| < \varepsilon}$$

and hence completes the proof .

Alternative definition of limit (2.5) : Page (53)

$$\lim_{x \to a} f(x) = L$$

means that for every $\varepsilon > 0$, there is a $\delta > 0$ such that if x is in the open interval

$(a-\delta, a+\delta)$ and $x \neq a$, then $f(x)$ is in the open interval $(L-\varepsilon, L+\varepsilon)$.

2.3 TECHNIQUES FOR FINDING LIMITS : Page (58)

Theorem (2.7) : Page (59)

(i) $\lim_{x \to a} c = c$. (ii) $\lim_{x \to a} x = a$.

Theorem (2.8) : Page (60)

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist , then

(i) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

(ii) $\lim_{x \to a} [f(x) . g(x)] = \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$.

(iii) $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$.

(iv) $\lim_{x \to a} [c f(x)] = c \left[\lim_{x \to a} f(x)\right]$.

(v) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$.

Theorem (2.9) : Page (60)

If m , b , and a are real numbers , then

$$\lim_{x \to a} (m x + b) = m a + b$$

Example (1) : Page (61)

Find $\lim\limits_{x \to 2} \dfrac{3x+4}{5x+7}$.

Solution

* $\lim\limits_{x \to 2} \dfrac{3x+4}{5x+7} = \dfrac{\lim\limits_{x \to 2}(3x+4)}{\lim\limits_{x \to 2}(5x+7)} = \dfrac{3(2)+4}{5(2)+7} = \boxed{\dfrac{10}{17}}$.

<u>Example (2) :</u> Page (61)

Prove that $\lim\limits_{x \to a} x^3 = a^3$.

Solution

* Since $\lim\limits_{x \to a} x = a$,

$$\lim_{x \to a} x^3 = \lim_{x \to a}(x \,.\, x \,.\, x)$$

$$= \left(\lim_{x \to a} x\right).\left(\lim_{x \to a} x\right).\left(\lim_{x \to a} x\right).$$

$$= a \,.\, a \,.\, a = \boxed{a^3}.$$

<u>Theorem (2.10) :</u> Page (62)

If n is a positive integer , then

(i) $\lim\limits_{x \to a} x^n = a^n$.

(ii) $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$, provided $\lim\limits_{x \to a} f(x)$ exists .

<u>Example (3) :</u> Page (62)

Find $\lim_{x \to 2} (3x+4)^5$.

Solution

$$* \lim_{x \to 2} (3x+4)^5 = \left[\lim_{x \to 2} (3x+4)\right]^5$$

$$= [3(2)+4]^5 = 10^5 = \boxed{100,000}.$$

Example (4) : Page (62)

Find $\lim_{x \to 2} \left(5x^3+3x^2-6\right)$.

Solution

$$\lim_{x \to -2} \left(5x^3+3x^2-6\right) = \lim_{x \to -2} \left(5x^3\right) + \lim_{x \to -2} \left(3x^2\right) - \lim_{x \to -2} (6)$$

$$= 5 \lim_{x \to -2} \left(x^3\right) + 3 \lim_{x \to -2} \left(x^2\right) - 6$$

$$= 5(-2)^3 + 3(-2)^2 - 6$$

$$= 5(-8) + 3(4) - 6 = \boxed{-34}.$$

Theorem (2.11) : Page (62)

If f is a polynomial function and a is a real number , then

$$\lim_{x \to a} f(x) = f(a).$$

Corollary (2.12) : Page (63)

If q is a rational function and a is in the domain of q , then

$$\lim_{x \to a} q(x) = q(a).$$

Example (5) : *Page (63)*

Find $\lim\limits_{x \to 3} \dfrac{5x^2 - 2x + 1}{4x^3 - 7}$.

Solution

$$* \lim_{x \to 3} \frac{5x^2 - 2x + 1}{4x^3 - 7} = \frac{5(3)^2 - 2(3) + 1}{4(3)^3 - 7}$$

$$= \frac{45 - 6 + 1}{108 - 7} = \boxed{\frac{40}{101}} .$$

Theorem (2.13) : *Page (63)*

If $a > 0$ and n is a positive integer , or if $a \leq 0$ and n is an odd positive integer , then

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} .$$

Example (6) : *Page (64)*

Find $\lim\limits_{x \to 8} \dfrac{x^{2/3} + 3\sqrt{x}}{4 - (16 / x)}$.

Solution

$$* \lim_{x \to 8} \frac{x^{2/3} + 3\sqrt{x}}{4 - (16 / x)} = \frac{\lim\limits_{x \to 8} \left(x^{2/3} + 3\sqrt{x}\right)}{\lim\limits_{x \to 8} \left[4 - (16 / x)\right]}$$

$$= \frac{\lim\limits_{x \to 8} x^{2/3} + \lim\limits_{x \to 8} 3\sqrt{x}}{\lim\limits_{x \to 8} 4 - \lim\limits_{x \to 8} (16 / x)}$$

$$= \frac{8^{2/3} + 3\sqrt{8}}{4 - (16 / 8)}$$

$$= \frac{4 + 4\sqrt{2}}{4 - 2} = \frac{4 + 4\sqrt{2}}{2} = \boxed{2 + 3\sqrt{2}}.$$

Theorem (2.14) : ***Page (64)***

If a function f has a limit as x approaches a, then

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)},$$

provided either n is an odd positive integer or n is an even positive integer and $\lim_{x \to a} f(x) > 0$.

Example (7) : ***Page (64)***

Find $\lim_{x \to 5} \sqrt[3]{3x^2 - 4x + 9}$.

Solution

$$* \lim_{x \to 5} \sqrt[3]{3x^2 - 4x + 9} = \sqrt[3]{\lim_{x \to 5} \left(3x^2 - 4x + 9\right)}$$

$$= \sqrt[3]{3(5)^2 - 4(5) + 9}$$

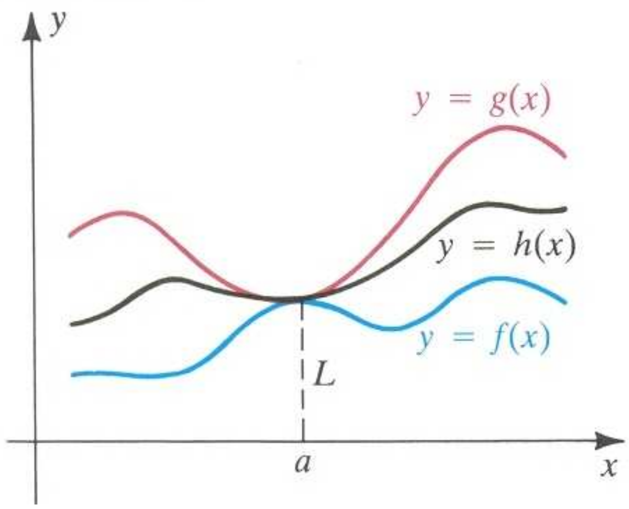
$$= \sqrt[3]{64} = \boxed{2 + 3\sqrt{2}}.$$

Theorem (2.15) : ***Page (64)***

Suppose $f(x) \le h(x) \le g(x)$ *for every x in an open interval containing a, except possibly at a,*

If $\lim_{x \to a} f(x) = L = \lim_{x \to a} g(x)$, *then* $\lim_{x \to a} h(x) = L$.

Figure 2.21



Example (8) : ***Page (65)***

Use the sandwich theorem to prove that $\lim\limits_{x \to 0} x^2 \sin \dfrac{1}{x^2} = 0$.

Solution

* *Since* $-1 \le \sin t \le 1$ *for every real number* t ,

$$-1 \le \sin \frac{1}{x^2} \le 1 \ , \quad \text{for every} \quad x \ne 0$$

* *Multiplying by* x^2 *(which is positive if* $x \ne 0$ *) , we obtain*

$$-x^2 \le x^2 \sin \frac{1}{x^2} \le x^2$$

* *This inequality implies that the graph of* $y = x^2 \sin \dfrac{1}{x^2}$ *lies between the parabolas* $y = -x^2$ *and* $y = x^2$.

* *Since* $\lim\limits_{x \to 0} \left(-x^2\right) = 0$, $\lim\limits_{x \to 0} \left(x^2\right) = 0$ *, then*

$$\boxed{\lim_{x \to 0} x^2 \sin \frac{1}{x^2} = 0} \ .$$

Home work

Exercises 2.1: 10,14,25,44 and 46.

Exercises 2.2: 15 and 16.

Exercises 2.3: 20- 25, 35, 36, 37 and 63

2.4 LIMITS INVOLVING INFINITY : *Page (68)*

Example (2) : *Page (70)*

Find each limit , if it exists .

(a) $\lim\limits_{x \to 4^-} \dfrac{1}{(x-4)^3}$. (b) $\lim\limits_{x \to 4^+} \dfrac{1}{(x-4)^3}$. (c) $\lim\limits_{x \to 4} \dfrac{1}{(x-4)^3}$.

Solution

(a) If x is close to 4 and $x < 4$, then $x - 4$ is close to 0 and negative , and

$$\lim_{x \to 4^-} \frac{1}{(x-4)^3} = \boxed{-\infty} .$$

(a) If x is close to 4 and $x > 4$, then $x - 4$ is close to 0 and positive , and

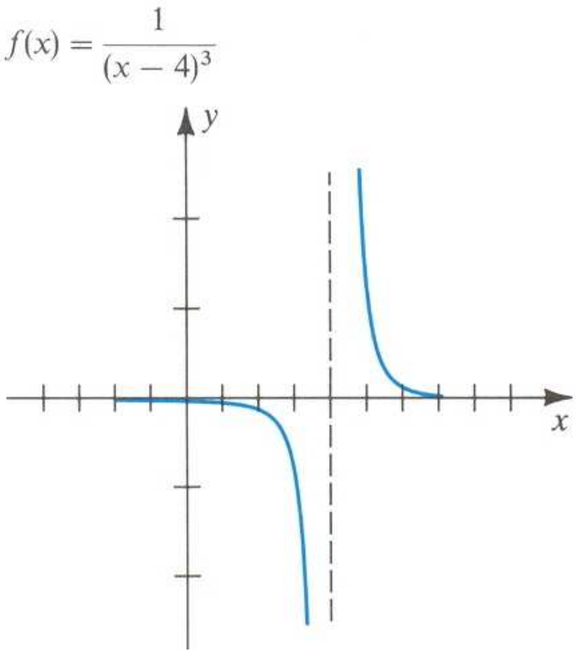
$$\lim_{x \to 4^+} \frac{1}{(x-4)^3} = \boxed{\infty} .$$

(c) Since $\lim\limits_{x \to 4^-} \dfrac{1}{(x-4)^3} \neq \lim\limits_{x \to 4^+} \dfrac{1}{(x-4)^3}$ *, then*

$$\boxed{\lim_{x \to 4} \frac{1}{(x-4)^3} \text{ does not exist}} .$$

** The graph of* $y = \dfrac{1}{(x-4)^3}$ *is sketched in Figure 2.29 . The line $x = 4$ is a vertical asymptote .*

Figure 2.29



Definition (2.16) : Page (71)

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$, there is a $M > 0$ such that

$$\text{If} \quad x > M \ , \quad \text{then} \quad |f(x) - L| < \varepsilon \, .$$

Definition (2.17) : Page (72)

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$, there is a $N < 0$ such that

$$\text{If} \quad x < N \ , \quad \text{then} \quad |f(x) - L| < \varepsilon \, .$$

Definition (2.18) : Page (73)

If k is a positive rational number and c is any number , then

$$\lim_{x \to \infty} \frac{c}{x^k} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{c}{x^k} = 0 \ ,$$

provided x^k is always defined .

Example (3) : *Page (73)*

Find $\lim\limits_{x \to -\infty} \dfrac{2x^2 - 5}{3x^2 + x + 2}$.

Solution

* *Since the highest power of x in the denominator is 2 , we first divide numerator and denominator by x^2 , obtaining*

$$\lim_{x \to -\infty} \frac{2x^2 - 5}{3x^2 + x + 2} = \lim_{x \to -\infty} \frac{\dfrac{2x^2 - 5}{x^2}}{\dfrac{3x^2 + x + 2}{x^2}}$$

$$= \lim_{x \to -\infty} \frac{2 - \dfrac{5}{x^2}}{3 + \dfrac{1}{x} + \dfrac{2}{x^2}}$$

$$= \frac{\lim\limits_{x \to -\infty}\left(2 - \dfrac{5}{x^2}\right)}{\lim\limits_{x \to -\infty}\left(3 + \dfrac{1}{x} + \dfrac{2}{x^2}\right)}$$

$$= \frac{\lim\limits_{x \to -\infty} 2 - \lim\limits_{x \to -\infty} \dfrac{5}{x^2}}{\lim\limits_{x \to -\infty} 3 + \lim\limits_{x \to -\infty} \dfrac{1}{x} + \lim\limits_{x \to -\infty} \dfrac{2}{x^2}}$$

$$= \frac{2 - 0}{3 + 0 + 0} = \boxed{\frac{2}{3}} .$$

Example (4) : *Page (74)*

Find $\lim\limits_{x \to \infty} \dfrac{2x^2 - 5}{3x^4 + x + 2}$.

Solution

* Since the highest power of x in the denominator is 4 , we first divide numerator and denominator by x^4 , obtaining

$$\lim_{x \to \infty} \frac{2x^2 - 5}{3x^4 + x + 2} = \lim_{x \to \infty} \frac{\dfrac{2}{x^2} - \dfrac{5}{x^4}}{3 + \dfrac{1}{x^3} + \dfrac{2}{x^4}}$$

$$= \frac{0 - 0}{3 + 0 + 0} = \frac{0}{3} = \boxed{0} .$$

Example (5) : Page (74)

Find $\lim\limits_{x \to \infty} \dfrac{2x^3 - 5}{3x^2 + x + 2}$.

Solution

* Since the highest power of x in the denominator is 2 , we first divide numerator and denominator by x^2 , obtaining

$$\lim_{x \to \infty} \frac{2x^2 - 5}{3x^2 + x + 2} = \lim_{x \to \infty} \frac{2x - \dfrac{5}{x^2}}{3 + \dfrac{1}{x} + \dfrac{2}{x^2}}$$

$$= \frac{\infty - 0}{3 + 0 + 0} = \frac{\infty}{3} = \boxed{\infty} .$$

Example (6) : Page (74)

If $f(x) = \dfrac{\sqrt{9x^2 + 2}}{4x + 3}$, find

(a) $\lim_{x \to \infty} f(x)$.   (b) $\lim_{x \to -\infty} f(x)$.

Solution

$$f(x) = \frac{\sqrt{9x^2 + 2}}{4x + 3}$$

(a) If x is large and positive, then

$$\sqrt{9x^2 + 2} \approx \sqrt{9x^2} = 3x \quad \text{and} \quad 4x + 3 \approx 4x$$

and hence

$$f(x) = \frac{\sqrt{9x^2 + 2}}{4x + 3} \approx \frac{3x}{4x} = \frac{3}{4}$$

this suggests that $\boxed{\lim_{x \to \infty} f(x) = \frac{3}{4}}$.

* *To give a rigorous proof we may write*

$$\lim_{x \to \infty} \frac{\sqrt{9x^2 + 2}}{4x + 3} = \lim_{x \to \infty} \frac{\sqrt{x^2\left(9 + \frac{2}{x^2}\right)}}{4x + 3}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x^2}\ \sqrt{9 + \frac{2}{x^2}}}{4x + 3}$$

If x is positive, then $\sqrt{x^2} = x$ *, and dividing numerator and denominator of the last fraction by x gives us*

$$\lim_{x \to \infty} \frac{\sqrt{9x^2 + 2}}{4x + 3} = \lim_{x \to \infty} \frac{\sqrt{9 + \frac{2}{x^2}}}{4 + \frac{3}{x}}$$

$$= \frac{\sqrt{9 + 0}}{4 + 0} = \boxed{\frac{3}{4}}.$$

(b) *If x is large negative, then $\sqrt{x^2} = -x$. If we use the same steps as in part (a), we obtain*

$$\lim_{x \to -\infty} \frac{\sqrt{9x^2+2}}{4x+3} = \lim_{x \to -\infty} \frac{\sqrt{x^2}\sqrt{x^2\left(9+\frac{2}{x^2}\right)}}{4x+3}$$

$$= \lim_{x \to -\infty} \frac{(-x)\sqrt{9+\frac{2}{x^2}}}{4x+3}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{9+\frac{2}{x^2}}}{4+\frac{3}{x}}$$

$$= \frac{-\sqrt{9+0}}{4+0} = \boxed{-\frac{3}{4}}.$$

Definition (2.19) : ***Page (76)***

$$\lim_{x \to a} f(x) = \infty$$

Means that for every $M > 0$, there is a $\delta > 0$ such that

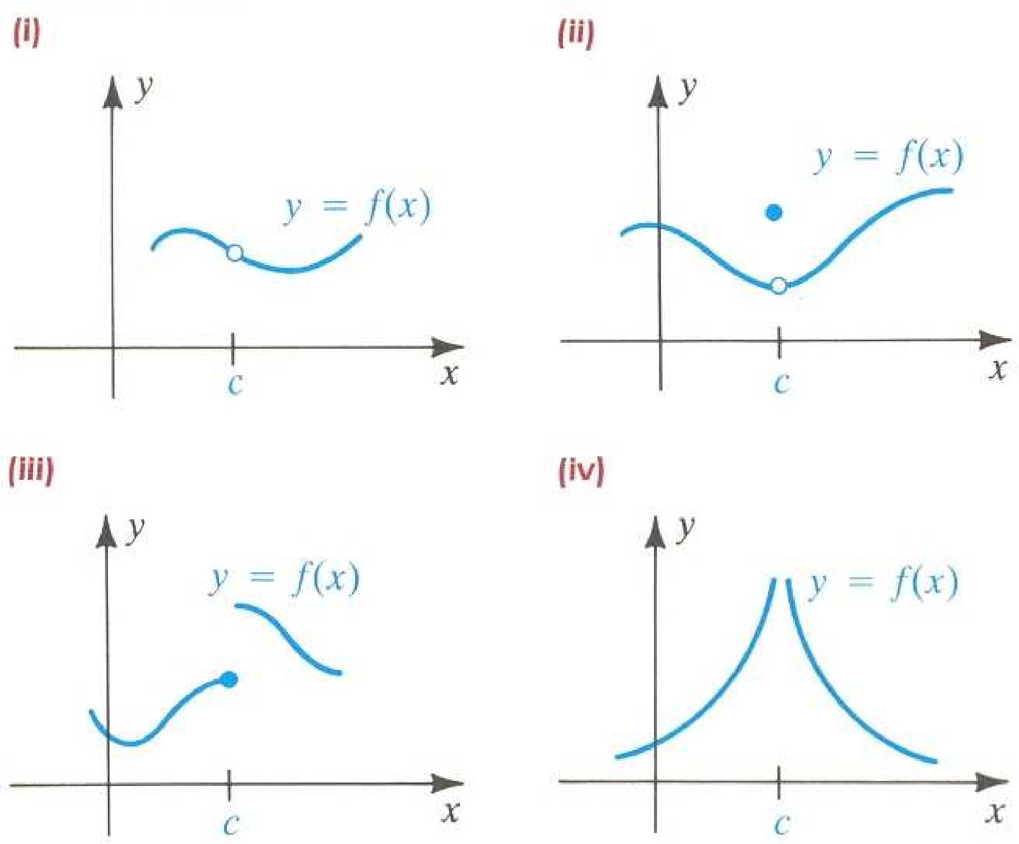
If $0 < |x-a| < \delta$, *then* $f(x) > M$.

Home Work

Exercises 2.4: 11, 13, 21 and 22.

2.5 CONTINUOUS FUNCTIONS : ***Page (77)***

Figure 2.35



**** Not that :***

In (i) of the Figure 2.35 , $f(c)$ *is not defined .*

In (ii) , $f(c)$ *is defined ;however ,* $\lim_{x \to c} f(x) \neq f(c)$ *.*

In (iii) , $\lim_{x \to c} f(x)$ *does not exist .*

In (iv) , $f(c)$ *is undefined and , in addition ,* $\lim_{x \to c} f(x) = \infty$ *.*

Definition (2.20) : ***Page (78)***

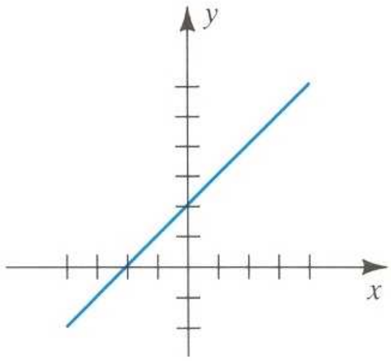
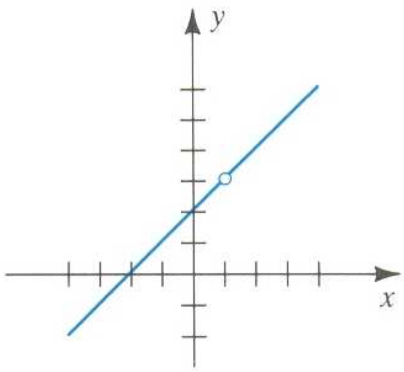
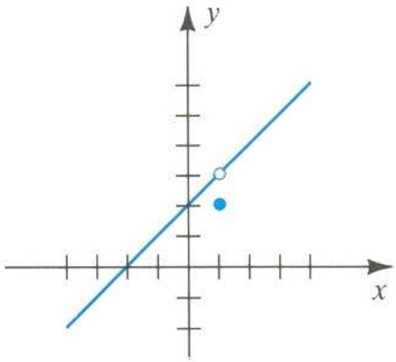
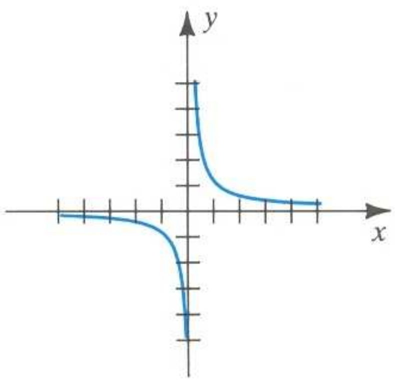
A function f *is continuous at a number* c *if the following conditions are satisfied :*

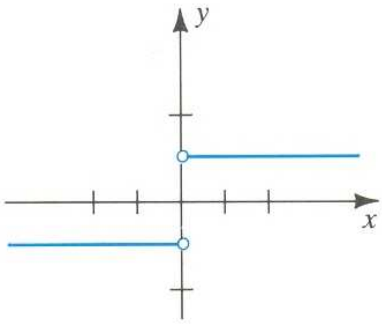
(i) $f(c)$ *is defined .*

(ii) $\lim_{x \to c} f(x)$ *exists .*

(iii) $\lim_{x \to c} f(x) = f(c)$ *.*

ILLUSTRATION : ***Page (79)***

FUNCTION VALUE	GRAPH	DISCONTINUITIES
$f(x) = x + 2$		None , since for every c, $\lim_{x \to c} f(x) = c + 2 = f(c)$
$g(x) = \dfrac{x^2 + x + 2}{x - 1}$		$c = 1$ since $g(1)$ is undefined (removable discontinuity).
$h(x) = \begin{cases} \dfrac{x^2 + x - 2}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$		$c = 1$ since $\lim_{x \to 1} h(x) = 3 \neq h(1)$ (removable discontinuity).
$h(x) = \dfrac{1}{x}$		$c = 0$ since $h(0)$ does not exist also $\lim_{x \to 0} h(x)$ does not exist (Infinite discontinuity).

$p(x)=\dfrac{\lvert x\rvert}{x}$		$c=0$ since $p(0)$ is undefined and also $\lim\limits_{x\to 0} p(x)$ does not exist (*jump discontinuity*).

Definition (2.21) : *Page (80)*

> *(i) A polynomial function f is continuous at every real number c.*
>
> *(ii) A rational function $q=\dfrac{f}{g}$ is continuous at every number except the numbers c such that $g(c)=0$.*

Example (2) : *Page (80)*

If $f(x)=\dfrac{x^2-1}{x^3+x^2-2x}$ *, find the discontinuities of f.*

Solution

$$f(x)=\frac{x^2-1}{x^3+x^2-2x}$$

* *Since f is a rational function , it follows that the only discontinuities at the zeros of the denominator* x^3+x^2-2x.

* *By factoring we obtain*

$$x^3+x^2-2x=x\left(x^2+x-2\right)=x\,(x+2)(x-1)$$

* *Setting each factor equal to zero , we see that the discontinuities of f are at* $\boxed{0\,,-2\,,\text{and } 1}$.

Definition (2.22) : *Page (81)*

> *Let a function f be defined on a closed interval $[a,b]$. The function f is continuous*

on $[a,b]$ if it is continuous on (a,b) and if , in addition ,

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b) .$$

Example (3) : Page (81)

If $f(x) = \sqrt{9 - x^2}$, sketch the graph of f and prove that f is continuous on the closed interval $[-3, 3]$.

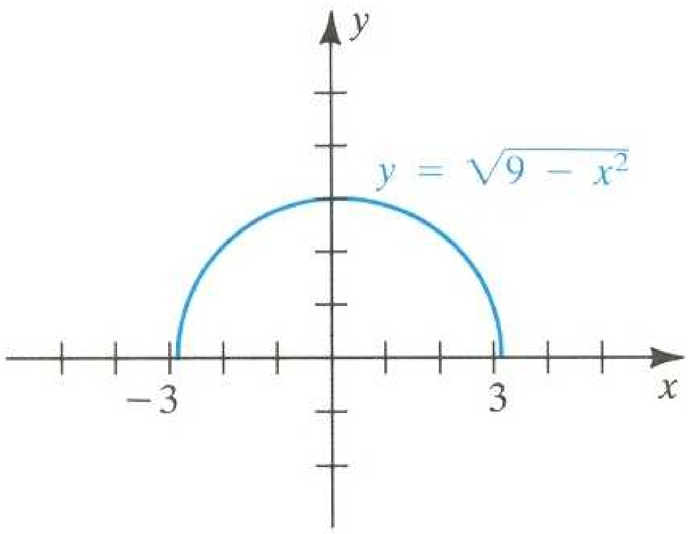
Solution

$$f(x) = \sqrt{9 - x^2}$$

* The graph of $x^2 + y^2 = 9$ is a circle with center at the origin and radius 3 .

Solving for y gives us $y = \pm\sqrt{9 - x^2}$, and hence the graph of $y = \sqrt{9 - x^2}$ is the upper half of that circle .

Figure 2.37



* If $-3 < c < 3$, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{c - x^2} = f(c) .$$

Hence f is continuous at c .

* All that remains is to check the endpoint of the interval $[-3, 3]$ using one-sided limits as follows :

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{9 - x^2} = 0 = f(-3)$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{9 - x^2} = 0 = f(3)$$

Thus , f is continuous from the right at -3 and from the left at 3 .

* *Then ,* $\boxed{f \text{ is continuous on } [-3, 3]}$.

Definition (2.23) : Page (82)

If f and g are continuous at c , then the following are also continuous at c :

(i) the sum $f + g$.

(ii) the difference $f - g$.

(iii) the product $f\,g$.

(iv) the quotient f / g , provided $g(c) \neq 0$.

Definition (2.24) : Page (83)

If $\lim_{x \to c} g(x) = b$ and if f is continuous at b , then

$$\lim_{x \to c} f(g(x)) = f(b) = f\left(\lim_{x \to c} g(x)\right).$$

Definition (2.25) : Page (83)

If g is continuous at c and if f is continuous at $b = g(c)$, then

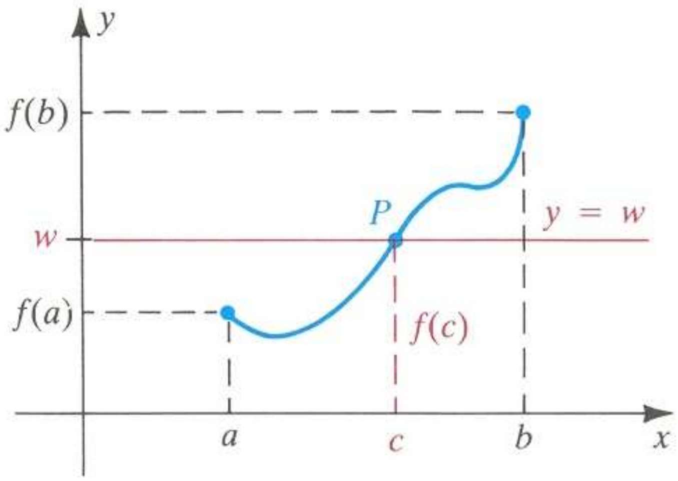
(i) $\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(g(c))$.

(ii) the composite function $f \circ g$ is continuous at c .

Intermediate value theorem (2.26) : Page (84)

If f is continuous on a closed interval $[a, b]$ and if w is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that .

Figure 2.38



Example (6) : *Page (84)*

Show that $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ **has a zero between 1 and 2 .**

Solution

$$f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$$

* Substituting 1 and 2 for x gives us the function values :

$$f(1) = 1 + 2 - 6 + 2 - 3 = -4$$

$$f(2) = 32 + 32 - 48 + 4 - 3 = 17$$

* Since $f(1)$ and $f(2)$ have opposite signs , it follows from the intermediate value theorem that $f(c) = 0$ for at least one real number c between 1 and 2 .

Home Worke

Exercises 2.5: 1-10, 31, 35 and 47.

CHAPTER (3)

THE DERIVATIVE

3.1 TANGENT LINES AND RATES OF CHANGE : Page (90)

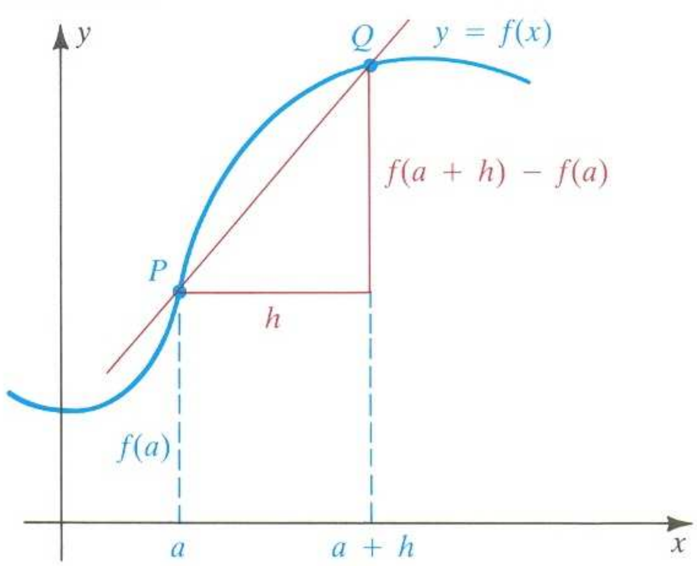
Definition (3.1) : Page (91)

The slope m_a of the tangent line to the graph of a function f at $P\left(a, f(a)\right)$ is

$$m_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Provided the limit exists .

Figure 3.4



Example (1) : Page (91)

Let $f(x) = x^2$, and let a be any real number .

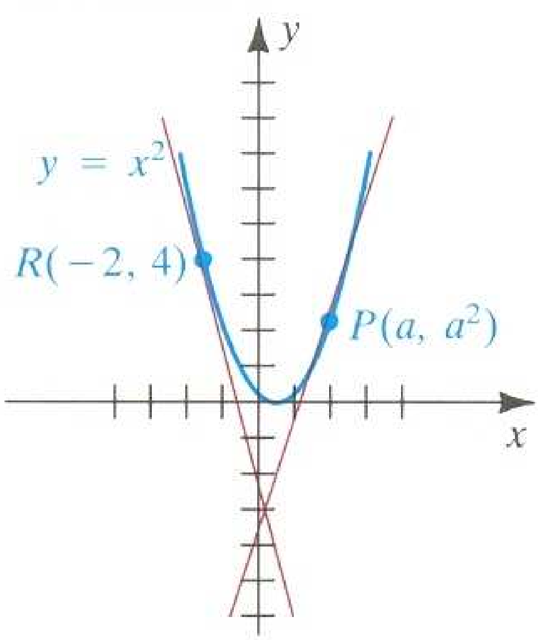
(a) Find the slope of the tangent line to the graph of f at $P\left(a, a^2\right)$.

(b) Find an equation of the tangent line at $R(-2, 4)$.

Solution

(a) The graph of $y = x^2$ and a typical point $P\left(a, a^2\right)$ are shown in Figure 3.5 .

Figure 3.5



* Applying Definition (3.1) , we see that the slope of the tangent line at P is

$$m_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - a^2}{h}$$

$$= \lim_{h \to 0} \frac{2ah + h^2}{h}$$

$$= \lim_{h \to 0} (2a + h) = \boxed{2a}.$$

(b) The slope of the tangent line at the point $R(-2, 4)$ is the special case of the formula $m_a = 2a$ with $a = -2$: that is ,

$$m_a = 2(-2) = -4$$

* Using the point-slope form $y - y_1 = m(x - x_1)$, we can express an equation for the tangent line as

$$y - 4 = -4(x - (-2))$$

$$y-4=-4x-8$$

$$\boxed{y=-4x-4}\,.$$

Definition (3.2) : Page (92)

The average velocity V_{av} of an object that travels a distance d in the time t is

$$V_{av}=\frac{d}{t}\,.$$

Definition (3.3) : Page (93)

Suppose a point P moves on a coordinate line l such that its coordinate at time t is $s(t)$. The velocity V_a of P at time a is

$$V_a=\lim_{h\to 0}\frac{s(a+h)-s(a)}{h}\,,$$

provided the limit exists .

Example (2) : Page (94)

A sandbag is dropped from a hot-air balloon that is hovering at a height of 512 feet above the ground . If air resistance is disregarded , then the distance $s(t)$ from the ground to the sandbag after t seconds is given by

$$s(t)=-16t^2+512$$

Find the velocity of the sandbag at

(a) $t=a$ sec . *(b)* $t=2$ sec .

(c) the instant it strikes the ground .

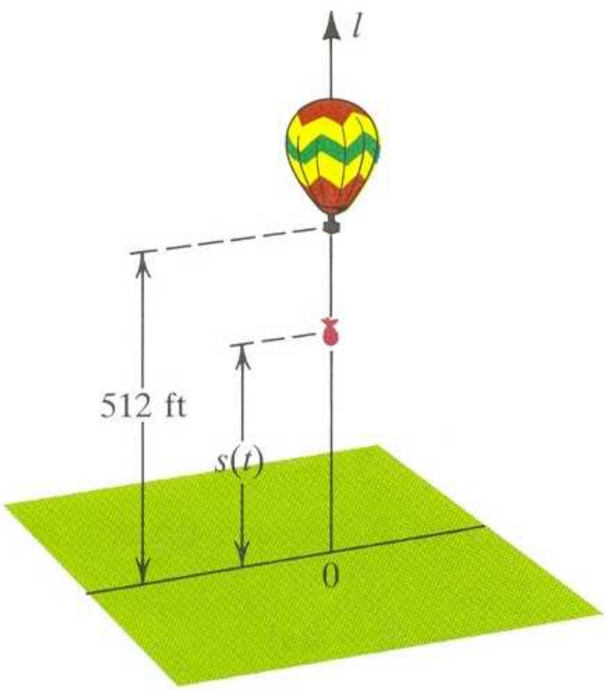
Solution

$$s(t)=-16t^2+512$$

(a) As shown in Figure 3.10 , we consider the sandbag to be moving along a vertical coordinate line l with origin at ground level . Note that at the instant it is dropped , $t=0$ and

$$s(0) = -16(0)^2 + 512 = 512 \ ft$$

Figure 3.10



*** To find the velocity of the sandbag at** $t = a$ **, we use Definition (3.3) , obtaining**

$$V_a = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

$$V_a = \lim_{h \to 0} \frac{\left[-16(a+h)^2 + 512\right] - \left(-16a^2 + 512\right)}{h}$$

$$V_a = \lim_{h \to 0} \frac{-16\left(a^2 + 2ah + h^2\right) + 512 + 16a^2 - 512}{h}$$

$$V_a = \lim_{h \to 0} \frac{-32ah - 16h^2}{h}$$

$$V_a = \lim_{h \to 0} \left(-32a - 16h\right) = \boxed{-32a \ ft / sec}.$$

*** The negative sign indicates that the motion of the sandbag is in the negative direction (downward) on** l **.**

(b) To find the velocity at $t=2$, we substitute 2 for a in the formula $V_a=-32a$, obtaining

$$V_2=-32(2)=\boxed{-64\ ft/sec}.$$

(c) The sandbag strikes the ground when the distance above the ground is zero-that is, when

$$s(t)=-16t^2+512=0$$

$$t^2=\frac{512}{16}=32$$

** This gives us*

$$t=\sqrt{32}=4\sqrt{2}\approx\boxed{5.7\ sec}.$$

Home work Exercises 3.1: 1, 7, 15.

3.2 DEFINITION OF DERIVATIVE : *Page (98)*

Definition (3.5) : *Page (98)*

The derivative of a function f is the function defined by

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Provided the limit exists .

Alternative definition of derivative (3.6) : *Page (99)*

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Applications of the derivative (3.7) : *Page (99)*

(i) Tangent line : The slope of the tangent line to the graph of $y=f(x)$ at the point $(a,f(a))$ is $f'(a)$.

(ii) Rate of change : If $y=f(x)$, the instantaneous rate of change of y with respect

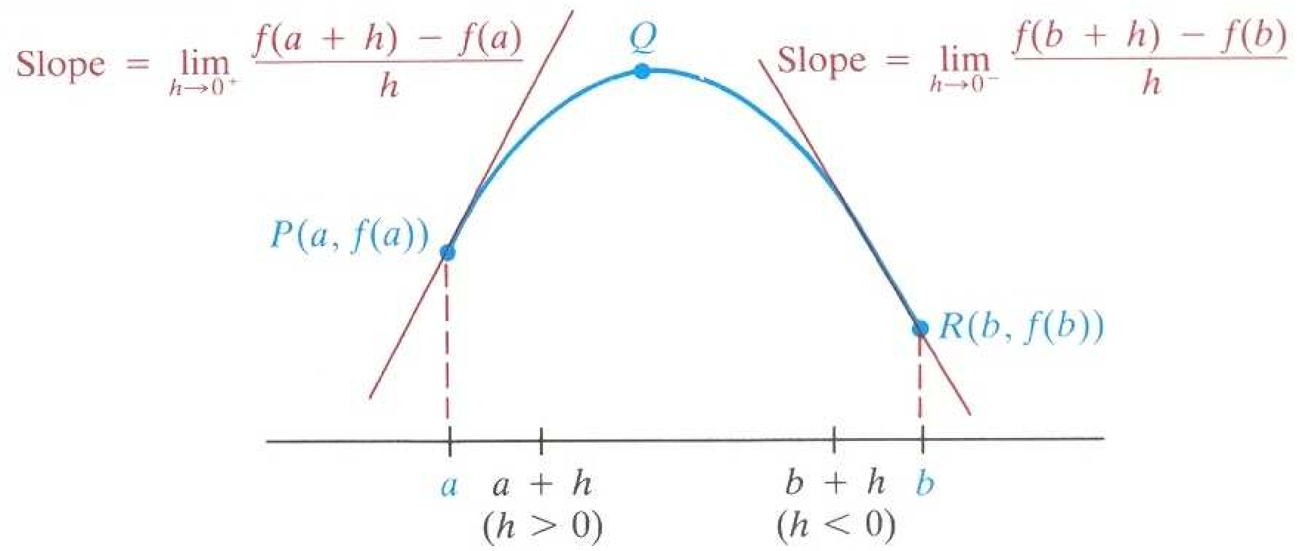
to x at a is $f'(a)$.

Definition (3.8) : *Page (99)*

A function f is differentiable on a closed interval $[a,b]$ if f is differentiable on the open interval (a,b) and if the following limits exist :

$$\lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h} \quad \text{and} \quad \lim_{h \to 0^-} \frac{f(b+h)-f(b)}{h}.$$

Figure 3.12

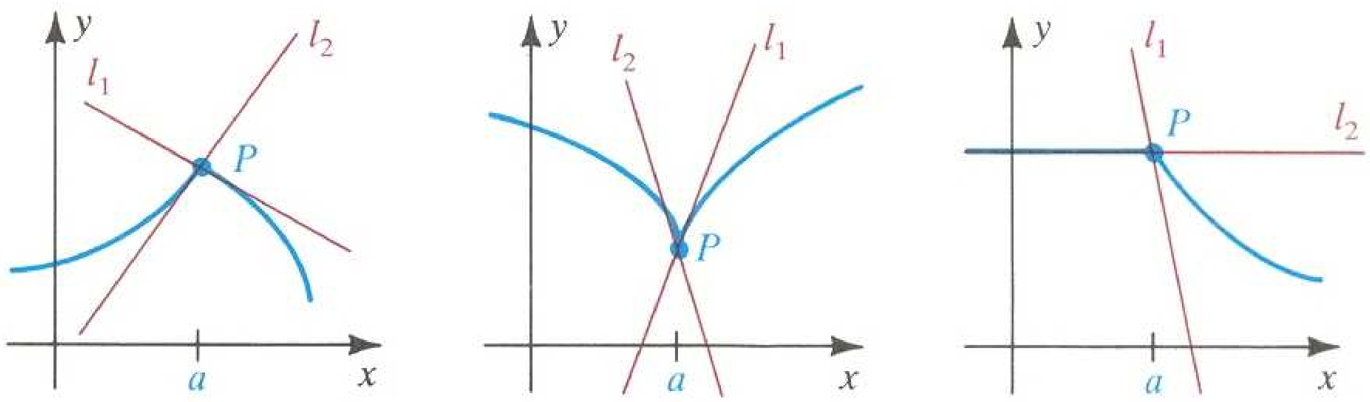


Definition (3.9) : *Page (100)*

The graph of a function f has a vertical tangent line $x = a$ at the point $P(a, f(a))$ if f is continuous at a and if

$$\lim_{x \to a} |f'(x)| = \infty .$$

Figure 3.13



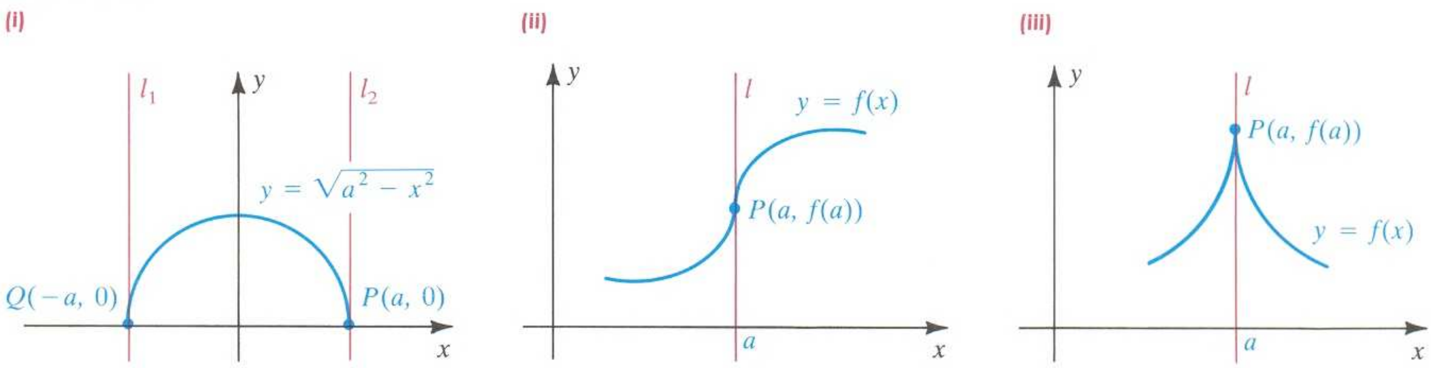
Definition (3.9) : ***Page (100)***

A point $P\left(a, f(a)\right)$ on the graph of a function f is a cusp if f is continuous at a and if the following two conditions hold :

(i) $f'(x) \to \infty$ as x approaches a from one side .

(ii) $f'(x) \to -\infty$ as x approaches a from the other side .

Figure 3.14



Example (1) : ***Page (101)***

If $f(x) = 3x^2 - 12x + 8$, find

(a) $f'(x)$. *(b) $f'(4)$, $f'(-2)$, and $f'(a)$.*

(c) the domain of f' .

Solution

$$f(x) = 3x^2 - 12x + 8$$

(a) By Definition (3.5) ,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left[3(x+h)^2-12(x+h)+8\right]-\left(3x^2-12x+8\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left(3x^2+6xh+3h^2-12x-12h+8\right)-\left(3x^2-12x+8\right)}{h}$$

$$=\lim_{h\to 0}\frac{6xh+3h^2-12h}{h}$$

$$=\lim_{h\to 0}\left(6x+3h-12\right)$$

$$=\boxed{6x-12}.$$

(b) Substituting for x in $f'(x)=6x-12$, we obtain

$$f'(4)=6(4)-12=\boxed{12}.$$

$$f'(-2)=6(-2)-12=\boxed{-24}.$$

$$f'(a)=\boxed{6a-12}.$$

(c) Since $f'(x)=6x-12$, the derivative exists for every real number x.

Hence the domain of f' is $\boxed{\mathbb{R}}$.

Example (2) : Page (101)

If $f(x)=3x^2-12x+8$, use the result of Example (1) to find

(a) the slope of the tangent line to the graph of this equation at the point $P(3,-1)$.

(b) the point on the graph at which the tangent line is horizontal.

Solution

$$f(x)=3x^2-12x+8$$

(a) If we let $f(x) = 3x^2 - 12x + 8$ *, then by (3.7) (i) and Example (1) , the slope of the tangent line at* $(x, f(x))$ *is* $f'(x) = 6x - 12$ *. In particular , the slope at* $P(3, -1)$ *is*

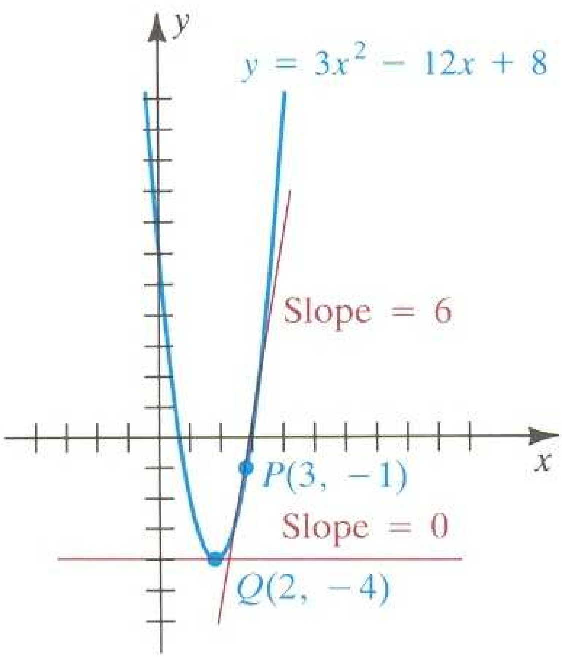
$$f'(3) = 6(3) - 12 = \boxed{6} .$$

(b) Since the tangent line is horizontal if the slope $f'(x)$ *is zero , we solve* $6x - 12 = 0$ *, obtaining* $x = 2$ *. The corresponding value of y is* -4 *.*

Hence the tangent line is horizontal at the point $\boxed{Q(2, -4)}$ *.*

** The graph of f (a parabola) and the tangent lines at P and Q are sketched in Figure 3.15 . Note the vertex of the parabola is the point* $Q(2, -4)$ *.*

Figure 3.15



Example (3) : Page (102)

If $f(x) = \sqrt{x}$,

(a) sketch the graph of f .

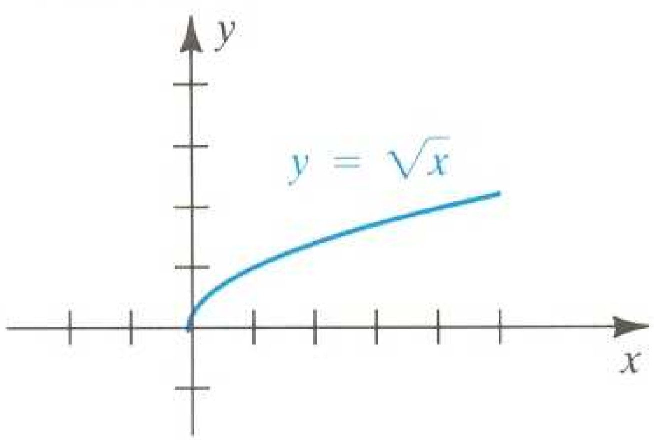
(b) find $f'(x)$ *and the domain of* f' *.*

Solution

$$f(x) = \sqrt{x}$$

(a) *The graph of f is sketched in Figure 3.16 . Note that the domain of f consists of all nonnegative numbers .*

Figure 3.16



(b) *Since $x=0$ is an endpoint of the domain of f , we shall examine the cases $x>0$ and $x=0$ separately .*

* *If $x>0$, then , by Definition (3.5) ,*

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

To find the limit , we first rationalize the numerator of the quotient and then simplify :

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{\sqrt{x}+\sqrt{x}}=\boxed{\frac{1}{2\sqrt{x}}}.$$

* *Since $x=0$ is an endpoint of the domain of f , we must use a one-sided limit to determine if $f'(0)$ exists . Using Definition (3.8) with $x=0$, we obtain*

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{\sqrt{0+h}-\sqrt{0}}{h}$$

$$= \lim_{h \to 0^+} \frac{\sqrt{h}}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt{h}} = \boxed{\infty}.$$

Since the limit does not exist , the domain of $f\,'$ *is*

$$\boxed{\textit{the set of positive real numbers}}\ .$$

The last limit shows that the graph of f *has a vertical tangent line (the y-axis) at the point* $(0\,,0\,)$.

Example (4) : ***Page (102)***

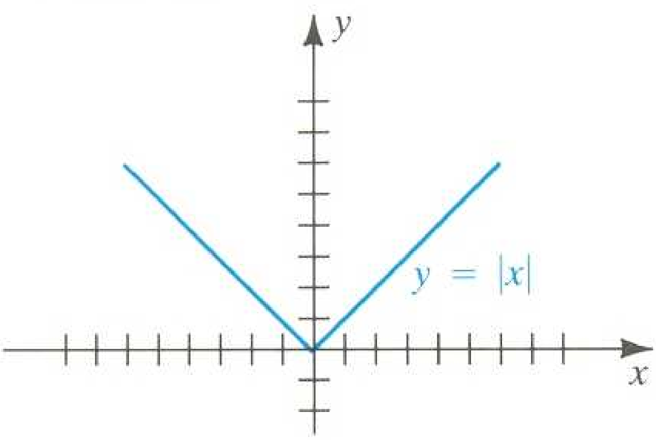
If $f\,(x\,) = \left|\,x\,\right|$ *, show that* f *is not differentiable at* 0 *.*

Solution

$$f\,(x\,) = \left|\,x\,\right|$$

* *The graph of* f *is sketched in Figure 3.17 .*

Figure 3.17



* *We can prove that* $f\,'\,(0\,)$ *does not exist by showing that the right-hand and left-hand derivatives are not equal . Using the limits in Definition (3.8) with* $a = 0$ *and* $b = 0$ *yields*

$$\lim_{h \to 0^+} \frac{f\,(0+h) - f\,(0\,)}{h} = \lim_{h \to 0^+} \frac{\left|0+h\right| - \left|0\right|}{h} = \lim_{h \to 0^+} \frac{\left|h\right|}{h} = \boxed{1}.$$

$$\lim_{h \to 0^-} \frac{f\,(0+h) - f\,(0\,)}{h} = \lim_{h \to 0^-} \frac{\left|0+h\right| - \left|0\right|}{h} = \lim_{h \to 0^-} \frac{\left|h\right|}{h} = \boxed{-1}.$$

Thus $f\,'\,(0\,)$ *does not exist , and hence*

$\boxed{f \text{ is not differentiable at } 0}$.

* Note that the graph of $y = |x|$ in Figure 3.17 has a corner and therefore no tangent line at point $P(0,0)$.

Theorem (3.11) : Page (103)

If a function f is differentiable at a , then f is continuous at a .

Power rule (3.14) : Page (104)

Let n be an integer

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$,

provide $x \neq 0$ when $n \leq 0$.

Theorem (3.15) : Page (106)

If $f(x) = c\,x^n$, then $f'(x) = (c\,n)\,x^{n-1}$.

Notation for the derivative of $y = f(x)$ (3.16) : Page (106)

$$f'(x) = D_x f(x) = D_x y = \frac{dy}{dx} = \frac{d}{dx} f(x) .$$

ILLUSTRATION : Page (106)

* $D_x\left(3x^7\right) = (3 . 7)\,x^6 = \boxed{21\,x^6}$.

* $D_t\left(\frac{1}{2} t^{12}\right) = \left(\frac{1}{2} . 12\right) x^{11} = \boxed{6\,x^{11}}$.

* $\frac{d}{dx}\left(4\,x^{3/2}\right) = \left(4 . \frac{3}{2}\right) x^{1/2} = \boxed{6\,x^{1/2}}$.

$$* \ \frac{d}{dr}\left(2r^{-4}\right)=2(-4)r^{-5}=\boxed{-\frac{8}{r^5}}.$$

$$* \ D_x\left(x^3\right)\Big]_{x=5}=3x^2\Big]_{x=5}=3\left(5^2\right)=\boxed{75}.$$

$$* \ \left[D_x\left(9x^{4/3}\right)\right]_{x=8}=\left[\left(9\cdot\frac{4}{3}\right)x^{1/3}\right]_{x=8}=12\left(8^{1/3}\right)=\boxed{24}.$$

__*Notation for higher derivatives (3.17) :*__ ***Page (107)***

$$f'(x),\ f''(x),\ f'''(x),\ f^{(4)}(x),\ \dots,\ f^{(n)}(x)\ .$$

$$D_x y\ ,\ D_x^2 y\ ,\ D_x^3 y\ ,\ D_x^4 y\ ,\ \dots,\ D_x^n y\ .$$

$$y'\ ,\ y''\ ,\ y'''\ ,\ y^{(4)}\ ,\ \dots,\ y^{(n)}\ .$$

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \frac{d^3y}{dx^3},\ \dots,\ \frac{d^ny}{dx^n}\ .$$

__*Example (5) :*__ ***Page (107)***

Find the first four derivatives of $f(x)=4x^{3/2}$.

Solution

$$f(x)=4x^{3/2}$$

**** We use (3.15) four times :***

$$f'(x)=\left(4\cdot\frac{3}{2}\right)x^{1/2}=\boxed{6x^{1/2}}.$$

$$f''(x)=\left(6\cdot\frac{1}{2}\right)x^{-1/2}=\boxed{3x^{-1/2}}.$$

$$f'''(x)=3\left(-\frac{1}{2}\right)x^{-3/2}=\boxed{-\frac{3}{2}x^{-3/2}}.$$

$$f^{(4)}(x)=\left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}=\boxed{\frac{9}{4}x^{-5/2}}.$$

Home Work: **Exercises 3.2:** 3, 9, 15, 21, 23, 25, 33 and 43.

3.3 TECHNIQUES OF DIFFERENTIATION : *Page (109)*

Theorem (3.18) : *Page (110)*

(i) $D_x\, c = 0$.

(ii) $D_x(mx+b)=m$.

(iii) $D_x\left(x^n\right)=nx^{n-1}$.

(iv) $D_x\left[c\,f(x)\right]=c\,D_x\,f(x)$.

(v) $D_x\left[f(x)+g(x)\right]=D_x\,f(x)+D_x\,g(x)$.

(vi) $D_x\left[f(x)-g(x)\right]=D_x\,f(x)-D_x\,g(x)$.

Example (1) : *Page (111)*

If $f(x)=2x^4-5x^3+x^2-4x+1$ *, find* $f'(x)$.

Solution

$$f(x)=2x^4-5x^3+x^2-4x+1$$

$$f'(x)=D_x\left(2x^4-5x^3+x^2-4x+1\right)$$

$$=D_x\left(2x^4\right)-D_x\left(5x^3\right)-D_x\left(x^2\right)+D_x(1)$$

Remember that :

$$*\; D_x x^n = n\,x^{n-1}$$

$$= \boxed{8x^3 - 15x^2 + 2x - 4}.$$

Example (2) : Page (111)

If $y = 6\sqrt[3]{x^2} - \dfrac{4}{\sqrt{x}}$ at $P(1, 2)$.

Solution

$$y = 6\sqrt[3]{x^2} - \frac{4}{\sqrt{x}}$$

* We first express y in terms of rational exponents and then find dy/dx :

$$y = 6x^{2/3} - 4x^{-1/2}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(6x^{2/3}\right) - \frac{d}{dx}\left(4x^{-1/2}\right)$$

Remember that :

* $\boxed{D_x x^n = n x^{n-1}}$

$$= 6\left(\frac{2}{3}\right)x^{-1/3} - 4\left(-\frac{1}{2}\right)x^{-3/2}$$

$$= \frac{4}{x^{1/3}} + \frac{2}{x^{3/2}}$$

* To find the slope of the tangent line at $P(1, 2)$, we evaluate dy/dx at $x = 1$:

$$\left.\frac{dy}{dx}\right]_{x=1} = \frac{4}{1} + \frac{2}{1} = 6$$

* Using the point-slope form , we can express an equation of the tangent line as

$$y - y_1 = m\left(x - x_1\right)$$

$$y - 2 = 6\left(x - 1\right)$$

$$\boxed{6x - y = 4}.$$

Product rule (3.19) : *Page (112)*

$$D_x f(x)g(x) = f(x)\, D_x g(x) + g(x)\, D_x f(x) .$$

Example (3) : *Page (112)*

If $y = \left(x^3 + 1\right)\left(2x^2 + 8x - 5\right)$ *, find* $D_x y$.

Solution

$$y = \left(x^3 + 1\right)\left(2x^2 + 8x - 5\right)$$

** Using the product rule (3.19) , we have*

Remember that :

$$* \; D_x\left[f(x)g(x)\right] = f(x)D_x g(x) + g(x)D_x f(x)$$

$$D_x y = \left(x^3 + 1\right) D_x\left(2x^2 + 8x - 5\right) + \left(2x^2 + 8x - 5\right) D_x\left(x^3 + 1\right)$$

$$= \left(x^3 + 1\right)(4x - 8) + \left(2x^2 + 8x - 5\right)\left(3x^2\right)$$

$$= \left(4x^4 + 8x^3 + 4x + 8\right) + \left(6x^4 + 24x^3 - 15x^2\right)$$

$$= \boxed{10x^4 + 32x^3 - 15x^2 + 4x + 8} .$$

Example (4) : *Page (113)*

If $f(x) = x^{1/3}\left(x^2 - 3x + 2\right)$*, find*

(a) $f'(x)$.

(b) the x-coordinate of the points on the graph of f *at which the tangent line is either horizontal or vertical .*

Solution

$$f(x) = x^{1/3}\left(x^2 - 3x + 2\right)$$

(a) By the product rule (3.19) ,

Remember that :

$$* \quad D_x\left[f(x)g(x)\right]=f(x)D_x g(x)+g(x)D_x f(x)$$

$$f'(x)=x^{1/3}D_x\left(x^2-3x+2\right)+\left(x^2-3x+2\right)D_x x^{1/3}$$

$$=x^{1/3}\left(2x^2-3\right)+\left(x^2-3x+2\right)\left(\frac{1}{3}x^{-2/3}\right)$$

$$=\frac{3x\left(2x^2-3\right)+\left(x^2-3x+2\right)}{3x^{2/3}}$$

$$=\boxed{\frac{7x^2-12x+2}{3x^{2/3}}}.$$

(b) The tangent line to the graph of f is horizontal if its slope is zero . Setting $f'(x)=0$ and using the quadratic formula , we obtain

$$x=\frac{12\pm\sqrt{144-56}}{2(7)}=\frac{12\pm\sqrt{88}}{14}=\boxed{\frac{6\pm\sqrt{22}}{7}}.$$

** Referring to $f'(x)$, we see that the denominator $3x^{2/3}$ is zero at $x=0$. Since f is continuous at 0 and $\lim_{x\to 0}\left|f'(x)\right|=\infty$, it follows from Definition (3.9) that the graph of f has a vertical tangent line at $\boxed{x=0}$ - that is , the point $(0,0)$ (the origin) .*

Quotient rule (3.20) : *Page (113)*

$$D_x\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\,D_x f(x)-f(x)\,D_x g(x)}{\left[g(x)\right]^2}.$$

Example (5) : *Page (114)*

Find $\dfrac{dy}{dx}$ *if* $y=\dfrac{3x^2-x+2}{4x^2+5}$.

Solution

$$y=\frac{3x^2-x+2}{4x^2+5}$$

** By the quotient rule (3.20) ,*

Remember that :

$$* \quad D_x\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)D_xf(x)-f(x)D_xg(x)}{[g(x)]^2}$$

$$\frac{dy}{dx}=\frac{(4x^2+5)D_x(3x^2-x+2)-(3x^2-x+2)D_x(4x^2+5)}{(4x^2+5)^2}$$

$$=\frac{(4x^2+5)(6x-1)-(3x^2-x+2)(8x)}{(4x^2+5)^2}$$

$$=\frac{(24x^2-4x^2+30x-5)-(24x^3-8x^2+16x)}{(4x^2+5)^2}$$

$$=\boxed{\frac{4x^2+14x-5}{(4x^2+5)^2}} .$$

Reciprocal rule (3.21) : *Page (114)*

$$\boxed{D_x\left[\frac{1}{g(x)}\right]=-\frac{D_x\,g(x)}{[g(x)]^2}} .$$

Home Work: Exercises 3.3: 5, 9, 19, 21, 33, 35, 41, 53, 55, 73.

3.4 DERIVARIVE OF THE TRIGONOMETRIC FUNCTIONS :

Page (98)

Definition (3.22) : *Page (118)*

(i) $\lim\limits_{\theta \to 0} \sin \theta = 0$. (ii) $\lim\limits_{\theta \to 0} \sin \theta = 1$.

Definition (3.23) : *Page (119)*

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 .$$

Definition (3.24) : *Page (120)*

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0 .$$

Derivatives of the trigonometric functions (3.25) : *Page (121)*

$D_x \sin x = \cos x$. $D_x \cos x = -\sin x$.

$D_x \tan x = \sec^2 x$. $D_x \cot x = -\csc^2 x$.

$D_x \sec x = \sec x \tan x$. $D_x \csc x = -\csc x \cot x$.

Example (1) : *Page (122)*

Find y' *if* $y = \dfrac{\sin x}{1 + \cos x}$.

Solution

$$y = \frac{\sin x}{1 + \cos x}$$

** By the quotient rule (3.20) and (3.25) ,*

Remember that :

$$* \boxed{D_x\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)D_xf(x)-f(x)D_xg(x)}{[g(x)]^2}}$$

$$y'=\frac{(1+\cos x)(D_x\sin x)-(\sin x)\,D_x(1+\cos x)}{(1+\cos x)^2}$$

Remember that :

$$* \boxed{D_x\sin x=\cos x \quad , \quad D_x\cos x=-\sin x}$$

$$=\frac{(1+\cos x)(\cos x)-(\sin x)(0-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}$$

$$=\frac{\cos x+1}{(1+\cos x)^2}$$

$$=\boxed{\frac{1}{1+\cos x}}.$$

Example (2) : Page (123)

Find $g'(x)$ if $g(x)=\sec x\tan x$.

Solution

$$g(x)=\sec x\tan x$$

* By the product rule (3.19) and (3.25) ,

Remember that :

$$* \boxed{D_x[f(x)g(x)]=f(x)D_xg(x)+g(x)D_xf(x)}$$

$$g'(x)=(\sec x)(D_x\tan x)+(\tan x)(D_x\sec x)$$

Remember that :

$$* \quad D_x \tan x = \sec^2 x \quad , \quad D_x \sec x = \sec x \tan x$$

$$= (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x$$

$$= \sec x \left(\sec^2 x + \tan^2 x\right)$$

Remember that :

$$* \quad \begin{array}{l} \cos^2 x + \sin^2 x = 1 \\ 1 + \tan^2 x = \sec^2 x \end{array}$$

$$g'(x) = \boxed{\sec x \left(2\tan^2 x + 1\right)} ,$$

or

$$g'(x) = \boxed{\sec x \left(2\sec^2 x - 1\right)} .$$

Example (3) : *Page (123)*

Find $dy / d\theta$ **if** $y = \sec\theta \tan\theta$.

Solution

$$y = \sec\theta \tan\theta$$

* *We could use the product rule (3.19) as in Example (2) ; however , it is simpler to first change the form of* y *by using fundamental identities as follows :*

Remember that :

$$* \quad \sec\theta = \frac{1}{\cos\theta} \quad , \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$y = \sec\theta \tan\theta = \frac{1}{\cos\theta}\,\frac{\cos\theta}{\sin\theta} = \csc\theta$$

* *Applying (3.25) yields*

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\csc\theta = \boxed{-\csc\theta \cot\theta} .$$

Example (4) : ***Page (123)***

(a) Find the slope of the tangent lines to the graph of $y = \sin x$ *at the points with x-coordinates* $0, \pi/3, \pi/2, 2\pi/3$, *and* π .

(b) Sketch the graph of $y = \sin x$ *and the tangent lines of part (a) .*

(c) For what values of x *is the tangent line horizontal ?*

Solution

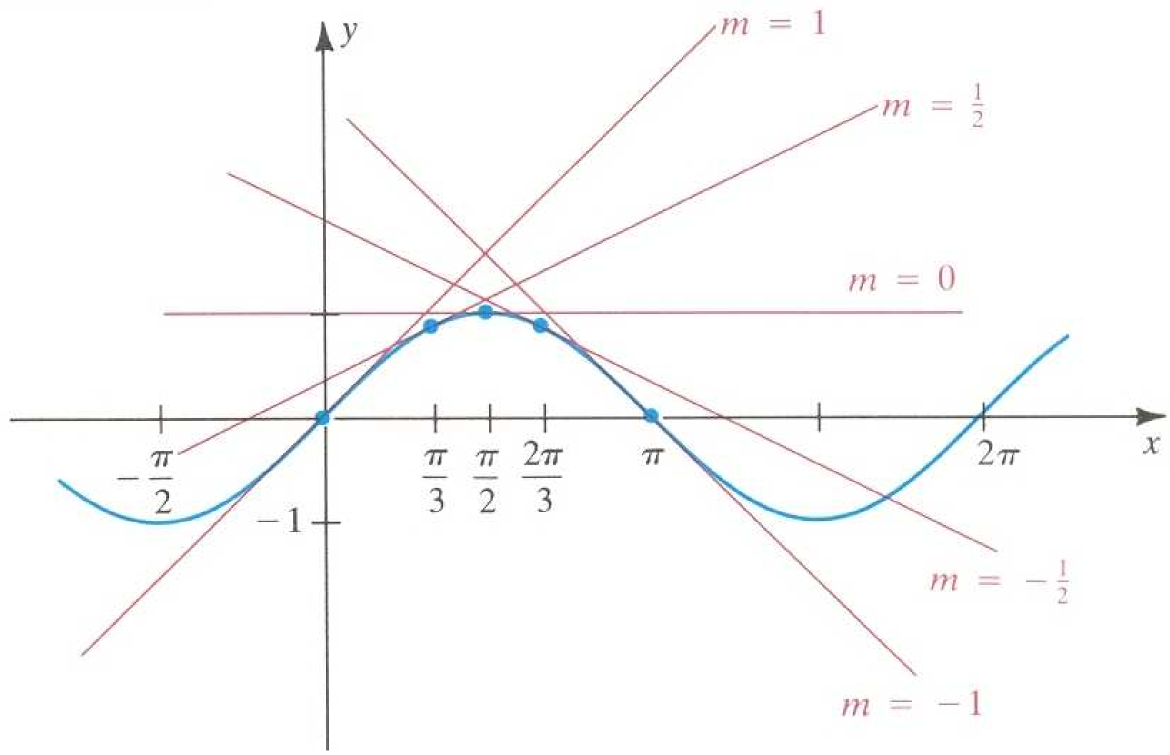
$$y = \sin x$$

(a) The slope of the tangent line at the point (x, y) *on the graph of the equation* $y = \sin x$ *is given by the derivative* $y' = \cos x$. *The slopes at the desired points are listed in the table on the following page .*

x	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
$y' = \cos x$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1

(b) A portion of the graph of $y = \sin x$ *and the tangent lines of part (a) are sketched in Figure 3.22.*

Figure 3.22



(c) A tangent line is horizontal if its slope is zero. Since the slope of the tangent line at the point (x, y) is y', we must solve the equation

$$y' = 0 \; ; \quad \text{that is} \; , \quad \cos x = 0 \; .$$

Thus tangent line is horizontal if $x = \pm\pi/2$, $x = \pm 3\pi/2$, and, in general, if

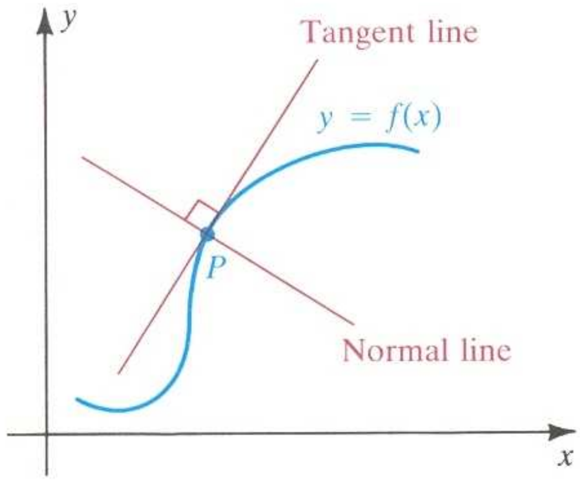
$$\boxed{x = (\pi/2) + \pi n \quad \text{for any integer } n} \; .$$

Note : *Page (123)*

* *If m is the slope of tangent line, then m_1 is the slope of normal line defined by*

$$\boxed{m_1 = \frac{-1}{m}}$$

Figure 3.23



Example (5) : *Page (124)*

Find an equation of the normal line to the graph of $y = \tan\theta$ at the point $P(\pi/4, 1)$, and illustrate it graphically.

Solution

$$y = \tan\theta$$

* *Since $y' = \sec^2\theta$, the slope m of the tangent line at P is*

$$m = \sec^2(\pi/4) = \left(\sqrt{2}\right)^2 = 2$$

And hence the slope of the normal line is

$$m_1 = -1 / m = -1 / 2 \quad .$$

* *Using the point-slope form , we can express an equation for the normal line as*

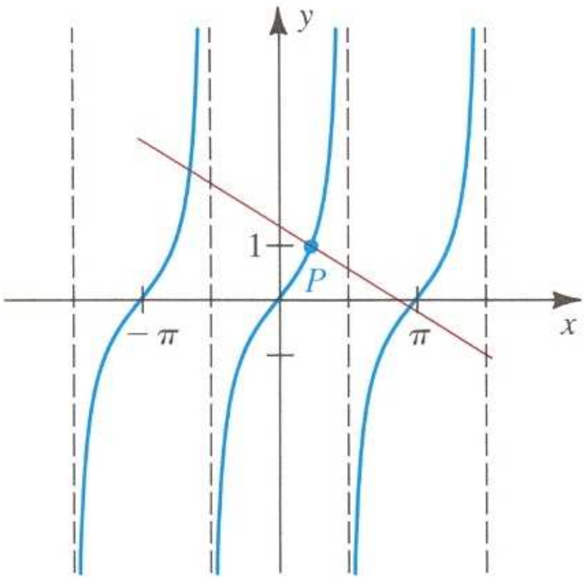
$$y - y_1 = m_1 (x - x_1)$$

$$y - 1 = -\frac{1}{2}\left(x - \frac{\pi}{4}\right)$$

or $$\boxed{y = -\frac{1}{2}x + \frac{\pi}{8} + 1} \ .$$

* *The graph of* $y = \tan\theta$ *for* $-3\pi / 2 < x < 3\pi / 2$ *and the normal line at* ***P*** *are sketched in Figure 3.24 .*

Figure 3.24



Home Work Exercises 3.4: 3,7, 15, 27, 29, 31, 35, 41, 47.

3.6 THE CHAIN RULE : ***Page (137)***

Chain rule (3.33) : ***Page (138)***

If $y = f(u)$, $u = g(x)$, *and the derivative* dy / du *and* du / dx *both exist , then the composite function defined by* $y = f(g(x))$ *has a derivative given by*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)\, g'(x) = f'(g(x))\, g'(x) \ .$$

Example (1) : *Page (140)*

Find $\frac{dy}{dx}$ *if* $y = \sqrt{u}$ *and* $u = x^2 + 1$.

Solution

$$y = \sqrt{u} \text{ and } u = x^2 + 1$$

* *If we substitute* $x^2 + 1$ *for* u *in* $y = \sqrt{u} = u^{1/2}$ *, we obtain*

$$y = \sqrt{x^2 + 1} = \left(x^2 + 1\right)^{1/2}.$$

* *We cannot find* dy / dx *by using previous differentiation formula ; however , using the chain rule (3.33) , we have*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{2}u^{-1/2}\right)(2x) = \frac{x}{\sqrt{u}}$$

and hence

$$\frac{dy}{dx} = \boxed{\frac{x}{\sqrt{x^2 + 1}}}.$$

Chain rule (3.34) : *Page (140)*

If $y = u^n$ *and* $u = g(x)$ *, then*

$$D_x\left(u^n\right) = nu^{n-1} D_x u ,$$

or , equivalently ,

$$D_x\left[g(x)\right]^n = n\left[g(x)\right]^{n-1} D_x g(x).$$

Example (2) : *Page (140)*

Find $f'(x)$ *if* $f(x) = \left(x^5 - 4x + 8\right)^7$.

Solution

$$f(x) = \left(x^5 - 4x + 8\right)^7$$

* *Using the power rule (3.34) with* $u = x^5 - 4x + 8$ *and* $n = 7$ *yields*

$$f'(x) = D_x\left(x^5 - 4x + 8\right)^7$$

Remember that :

$$* \quad \boxed{D_x\left[g(x)\right]^n = n\left[g(x)\right]^{n-1} D_x g(x)}$$

$$= 7\left(x^5 - 4x + 8\right)^6 D_x\left(x^5 - 4x + 8\right)$$

$$= \boxed{7\left(x^5 - 4x + 8\right)^6\left(5x^4 - 4\right)}.$$

Example (3) : Page (140)

Find $\dfrac{dy}{dx}$ *if* $y = \dfrac{1}{\left(4x^2 + 6x - 7\right)^3}$.

Solution

$$y = \frac{1}{\left(4x^2 + 6x - 7\right)^3}$$

* *Writing* $y = \left(4x^2 + 6x - 7\right)^{-3}$ *and using the power rule (3.34) with* $u = 4x^2 + 6x - 7$ *and* $n = -3$ *, we have*

$$\frac{dy}{dx} = \frac{d}{dx}\left(4x^2 + 6x - 7\right)^{-3}$$

Remember that :

$$* \quad \boxed{D_x\left[g(x)\right]^n = n\left[g(x)\right]^{n-1} D_x g(x)}$$

$$= -3\left(4x^2 + 6x - 7\right)^{-4} \frac{d}{dx}\left(4x^2 + 6x - 7\right)$$

$$= -3\left(4x^2 + 6x - 7\right)^{-4}\left(8x + 6\right)$$

$$= \boxed{\frac{-6\,(4x+3)}{\left(4x^2+6x-7\right)^4}}\,.$$

Example (4) : Page (141)

Find $f'(x)$ if $f(x)=\sqrt[3]{5x^2-x+4}$.

Solution

$$f(x)=\sqrt[3]{5x^2-x+4}$$

* **Writing** $f(x)=\left(5x^2-x+4\right)^{1/3}$ **and using the power rule (3.34) with** $u=5x^2-x+4$ **and** $n=\dfrac{1}{3}$ **, we obtain**

Remember that :

$$* \quad \boxed{D_x\left[g(x)\right]^n = n\left[g(x)\right]^{n-1} D_x g(x)}$$

$$f'(x)=\left(\frac{1}{3}\right)\left(5x^2-x+4\right)^{-2/3} D_x\left(5x^2-x+4\right)$$

$$=\left(\frac{1}{3}\right)\left(5x^2-x+4\right)^{-2/3}\left(10x-1\right)$$

$$= \boxed{\frac{10x-1}{3\,\sqrt[3]{\left(5x^2-x+4\right)^2}}}\,.$$

Example (5) : Page (141)

Find $F'(z)$ if $F(z)=(2z+5)^3\,(3z-1)^4$.

Solution

$$F(z)=(2z+5)^3\,(3z-1)^4$$

* Using first the product rule , second the power rule , and then factoring the result gives us

Remember that :

* $D_x\left[f(x)g(x)\right]=f(x)D_xg(x)+g(x)D_xf(x)$
* $D_x\left[g(x)\right]^n=n\left[g(x)\right]^{n-1}D_xg(x)$

$$F'(z)=(2z+5)^3D_x(3z-1)^4+(3z-1)^4D_x(2z+5)^3$$

$$=(2z+5)^3.4(3z-1)^3(3)+(3z-1)^4.3(2z+5)^2(2)$$

$$=6(2z+5)^2(3z-1)^3\left[2(2z+5)+(3z-1)\right]$$

$$=\boxed{6(2z+5)^2(3z-1)^3(7z+9)}.$$

Example (6) : Page (141)

Find y' if $y=(3x+1)^6\sqrt{2x-5}$.

Solution

* Since $y=(3x+1)^6(2x-5)^{1/2}$, we have , by the product and power rules ,

Remember that :

* $D_x\left[f(x)g(x)\right]=f(x)D_xg(x)+g(x)D_xf(x)$
* $D_x\left[g(x)\right]^n=n\left[g(x)\right]^{n-1}D_xg(x)$

$$y'=(3x+1)^6\frac{1}{2}(2x-5)^{-1/2}(2)+(2x-5)^{1/2}6(3x+1)^5(3)$$

$$=\frac{(3x+1)^6}{\sqrt{2x-5}}+18(3x+1)^5\sqrt{2x-5}$$

$$=\frac{(3x+1)^6+18(3x+1)^5(2x-5)}{\sqrt{2x-5}}$$

$$= \boxed{\frac{(3x+1)^5\,(39x-89)}{\sqrt{2x-5}}}.$$

Example (7) : **Page (142)**

Find $f'(x)$ if $f(x)=\left(7x+\sqrt{x^2+6}\right)^4$.

Solution

$$f(x)=\left(7x+\sqrt{x^2+6}\right)^4$$

** Applying the power rule yields*

Remember that :

$$* \boxed{D_x\left[g(x)\right]^n = n\left[g(x)\right]^{n-1} D_x g(x)}$$

$$f'(x)=4\left(7x+\sqrt{x^2+6}\right)^3 D_x\left(7x+\sqrt{x^2+6}\right)$$

$$=4\left(7x+\sqrt{x^2+6}\right)^3\left[D_x(7x)+D_x\sqrt{x^2+6}\right]$$

** Again applying the power rule , we have*

$$D_x\sqrt{x^2+6}=D_x\left(x^2+6\right)^{1/2}$$

$$=\frac{1}{2}\left(x^2+6\right)^{-1/2} D_x\left(x^2+6\right)$$

$$=\frac{1}{2\sqrt{x^2+6}}(2x)=\frac{x}{\sqrt{x^2+6}}$$

** Therefore ,* $$f'(x)=\boxed{4\left(7x+\sqrt{x^2+6}\right)^3\left(7+\frac{x}{\sqrt{x^2+6}}\right)}.$$

Theorem (3.35) : **Page (142)**

If $u = g(x)$ and g is differentiable, then

$$D_x \sin u = (\cos u) D_x u \,. \qquad D_x \cos u = (-\sin u) D_x u \,.$$

$$D_x \tan u = \left(\sec^2 u\right) D_x u \,. \qquad D_x \cot u = \left(-\csc^2 u\right) D_x u \,.$$

$$D_x \sec u = (\sec u \tan u) D_x u \,. \quad D_x \csc u = (-\csc u \cot u) D_x u \,.$$

Example (8) : Page (142)

If $y = \cos\left(5x^3\right)$, find $D_x y$ and $D_x^2 y$.

Solution

$$y = \cos\left(5x^3\right)$$

* Using the formula for $D_x \cos u$ in Theorem (3.35) with $u = 5x^3$, we have

$$D_x y = D_x \cos u$$

Remember that :

$$* \; D_x \cos u = (-\sin u) D_x u$$

$$= \left[-\sin\left(5x^3\right)\right] D_x \left(5x^3\right)$$

$$= \left[-\sin\left(5x^3\right)\right] \left(15x^2\right)$$

$$= \boxed{-15x^2 \sin\left(5x^3\right)} \,.$$

* To find $D_x^2 y$, we differentiate $D_x y = -15x^2 \sin\left(5x^3\right)$. Using the product rule and Theorem (3.35) gives us

$$D_x^2 y = -15x^2 D_x \sin\left(5x^3\right) + \sin\left(5x^3\right) D_x \left(-15x^2\right)$$

Remember that :

$$* \; D_x \sin u = (\cos u) D_x u$$

$$= -15x^2\cos\left(5x^3\right)D_x\left(5x^3\right)+\left[\sin\left(5x^3\right)\right](-30x)$$

$$= -15x^2\left[\cos\left(5x^3\right)\right]\left(15x^2\right)-30x\sin\left(5x^3\right)$$

$$= \boxed{-225x^4\cos\left(5x^3\right)-30x\sin\left(5x^3\right)}.$$

Example (9) : *Page (142)*

Find $f'(x)$ if $f(x)=\tan^3 4x$.

Solution

* First note that $f(x)=\tan^3 4x=(\tan 4x)^3$. Applying the power rule with $u=\tan 4x$ and $n=3$ yields

Remember that :

* $\boxed{D_x[g(x)]^n=n[g(x)]^{n-1}D_x g(x)}$

$$f'(x)=3(\tan 4x)^2 D_x \tan 4x$$

* Next , by Theorem (3.35) ,

Remember that :

* $\boxed{D_x \tan u=\left(\sec^2 u\right)D_x u}$

$$f'(x)=\left(3\tan^2 4x\right)\left(\sec^2 4x\right)D_x(4x)$$

$$=\left(3\tan^2 4x\right)\left(\sec^2 4x\right)(4)$$

* Thus $f'(x)=\boxed{12\tan^2 4x\sec^2 4x}.$

Example (10) : *Page (142)*

Find y' if $y=\sqrt{\sin 6x}$.

Solution

* *Writing* $y = (\sin 6x)^{1/2}$ *and using the power rule, we obtain*

Remember that :

$$* \quad \boxed{D_x[g(x)]^n = n[g(x)]^{n-1} D_x g(x)}$$

$$y' = \frac{1}{2}(\sin 6x)^{-1/2} D_x \sin 6x$$

* *Next, by Theorem (3.35),*

Remember that :

$$* \quad \boxed{D_x \sin u = (\cos u) D_x u}$$

$$y' = \frac{1}{2}(\sin 6x)^{-1/2} \cos 6x \, D_x(6x)$$

$$= \frac{1}{2}(\sin 6x)^{-1/2} (\cos 6x)(6)$$

$$= \boxed{\frac{3\cos 6x}{\sqrt{\sin 6x}}}.$$

Home Work Exercises 3.4: 3, 9, 15, 29, 39, 43, 57, 67, 73, 81.

3.7 IMPLICIT DIFFERENTIATION : *Page (146)*

Example (2) : *Page (148)*

Assuming that the equation $y^4 + 3y - 4x^3 = 5x + 1$ *determines, implicitly, a differentiable function* f *such that* $y = f(x)$*, find its derivative.*

Solution

$$y^4 + 3y - 4x^3 = 5x + 1$$

* *We regard* y *as a symbol that denotes* $f(x)$ *and consider the equation as an identity for every* x *in the domain of* f*. Since derivatives of both sides are equal, we obtain the following :*

$$D_x\left(y^4+3y-4x^3\right)=D_x\left(5x+1\right)$$

$$D_x\left(y^4\right)+D_x\left(3y\right)-D_x\left(4x^3\right)=D_x\left(5x\right)+D_x\left(1\right)$$

$$4y^3y'+3y'-12x^2=5+0$$

* *We now solve for y', obtaining*

$$\left(4y^3+3\right)y'=12x^2+5$$

or

$$y'=\frac{12x^2+5}{4y^3+3}$$

provided $4y^3+3\neq 0$.

* *Thus, if* $y=f(x)$, *then*

$$f'(x)=\boxed{\frac{12x^2+5}{4\left(f(x)\right)^3+3}}.$$

Example (3) : *Page (148)*

Find the slope of the tangent line to the graph of $y^4+3y-4x^3=5x+1$ *at the point* $P(1,-2)$.

Solution

$$y^4+3y-4x^3=5x+1$$

* *The slope of the tangent line at* $P(1,-2)$ *is the value of the derivative* y' *when* $x=1$ *and* $y=-2$.

* *The given equation is the same as that in Example (2), where we found that*

$$y'=\frac{12x^2+5}{4y^3+3}.$$

* *Substituting* 1 *for* x *and* -2 *for* y *gives us the following, where* $y'\big]_{(1,-2)}$ *denotes the value of* y' *when* $x=1$ *and* $y=-2$:

$$y'\Big]_{(1,-2)} = \frac{12(1)^2+5}{4(-2)^3+3} = \boxed{-\frac{17}{29}}\,.$$

Example (4) : Page (149)

If $y = f(x)$ *, where* f *is determined implicitly by the equation* $x^2 + y^2 = 1$ *, find* y'.

Solution

$$x^2 + y^2 = 1$$

* *Differentiating both sides of the equation with respect to* x *yields*

$$D_x\left(x^2 + y^2\right) = D_x(1)$$

$$D_x\left(x^2\right) + D_x\left(y^2\right) = D_x(1)$$

$$2x + 2y\,y' = 0$$

$$y\,y' = -x$$

$$y' = \boxed{-\frac{x}{y} \quad \text{if} \quad y \neq 0}\,.$$

Example (5) : Page (149)

Find y' *if* $4xy^3 - x^2y + x^3 - 5x + 6 = 0$.

Solution

$$4xy^3 - x^2y + x^3 - 5x + 6 = 0$$

* *Differentiating both sides of the equation with respect to* x *yields*

$$D_x\left(4xy^3\right) + D_x\left(x^2y\right) + D_x\left(x^3\right) - D_x(5x) + D_x(6) = D_x(0)$$

* *Since* y *denotes for some function* f *, the product rule must be applied to* $D_x\left(4xy^3\right)$ *and* $D_x\left(x^2y\right)$. *Thus ,*

Remember that :

$$* \boxed{D_x[f(x)g(x)] = f(x)D_x g(x) + g(x)D_x f(x)}$$

$$D_x(4xy^3) = 4x D_x(y^3) + y^3 D_x(4x)$$

$$= 4x(3y^2 y') + y^3(4)$$

$$= 12xy^2 y' + 4y^3$$

and $$D_x(x^2 y) = x^2 D_x(y) + y D_x(x^2)$$

$$= x^2 y' + y(2x)$$

** Substituting these expressions in the first equation of the solution and differentiating the other terms leads to*

$$(12xy^2 y' + 4y^3) - (x^2 y' + 2xy) + 3x^2 - 5 = 0$$

** Collecting the terms containing y' and transposing the remaining terms to the right side of the equation gives us*

$$(12xy^2 - x^2)y' = 5 - 3x^2 + 2xy - 4y^3$$

** Consequently ,* $$y' = \boxed{\frac{5 - 3x^2 + 2xy - 4y^3}{12xy^2 - x^2}},$$

provided $12xy^2 - x^2 \neq 0$.

Example (6) : ***Page (150)***

Find y' if $y = x^2 \sin y$.

Solution

$$y = x^2 \sin y$$

** Differentiating both sides of the equation with respect to x and using the product rule , we obtain*

Remember that :

$$* \boxed{D_x[f(x)g(x)] = f(x)D_x g(x) + g(x)D_x f(x)}$$

$$D_x y = x^2 D_x(\sin y) + (\sin y) D_x(x^2)$$

* Since $y = f(x)$ for some (implicit) function f, we have, by Theorem (3.35),

Remember that :

* $\boxed{D_x \sin u = (\cos u) D_x u}$

$$D_x \sin y = \cos y \; D_x y$$

* Using this equation and the fact that $D_x(x^2) = 2x$, we may rewrite the first equation of our solution as

$$D_x y = (x^2 \cos y) D_x y + (\sin y)(2x),$$

or $$y' = (x^2 \cos y) y' + 2x \sin y .$$

* Finally, we solve for y' as follows :

$$y' - (x^2 \cos y) y' = 2x \sin y$$

$$(1 - x^2 \cos y) y' = 2x \sin y$$

$$y' = \boxed{\frac{2x \sin y}{1 - x^2 \cos y}},$$

provided $1 - x^2 \cos y \neq 0$.

Example (7) : Page (150)

Find y'' if $y^4 + 3y - 4x^3 = 5x + 1$.

Solution

$$y^4 + 3y - 4x^3 = 5x + 1$$

* The equation was considered in Example (2), where we found that

$$y' = \frac{12x^2 + 5}{4y^3 + 3} .$$

** Hence* $$y'' = D_x\left(y'\right) = D_x\left(\frac{12x^2+5}{4y^3+3}\right)$$

** We now use the quotient rule, differentiating implicitly as follows :*

Remember that :

$$*\quad D_x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)D_x f(x) - f(x)D_x g(x)}{\left[g(x)\right]^2}$$

$$y'' = \frac{\left(4y^3+3\right)D_x\left(12x^2+5\right) - \left(12x^2+5\right)D_x\left(4y^3+3\right)}{\left(4y^3+3\right)^2}$$

$$= \frac{\left(4y^3+3\right)\left(24x\right) - \left(12x^2+5\right)\left(12y^2y'\right)}{\left(4y^3+3\right)^2}$$

** Substituting for* y' *yields*

$$y'' = \frac{\left(4y^3+3\right)\left(24x\right) - \left(12x^2+5\right).12y^2\left(\frac{12x^2+5}{4y^3+3}\right)}{\left(4y^3+3\right)^2}$$

$$= \boxed{\frac{\left(4y^3+3\right)^2\left(24x\right) - 12y^2\left(12x^2+5\right)}{\left(4y^3+3\right)^3}},$$

provided $4y^3+3 \neq 0$.

Home Work :Exercises 3.7: 1, 4, 7, 8, 15.

CHAPTER (4)

APPLICATIONS OF THE DERIVATIVE

4.1 EXTREMA OF FUNCTIONS : Page (166)

Definition (4.1) : Page (166)

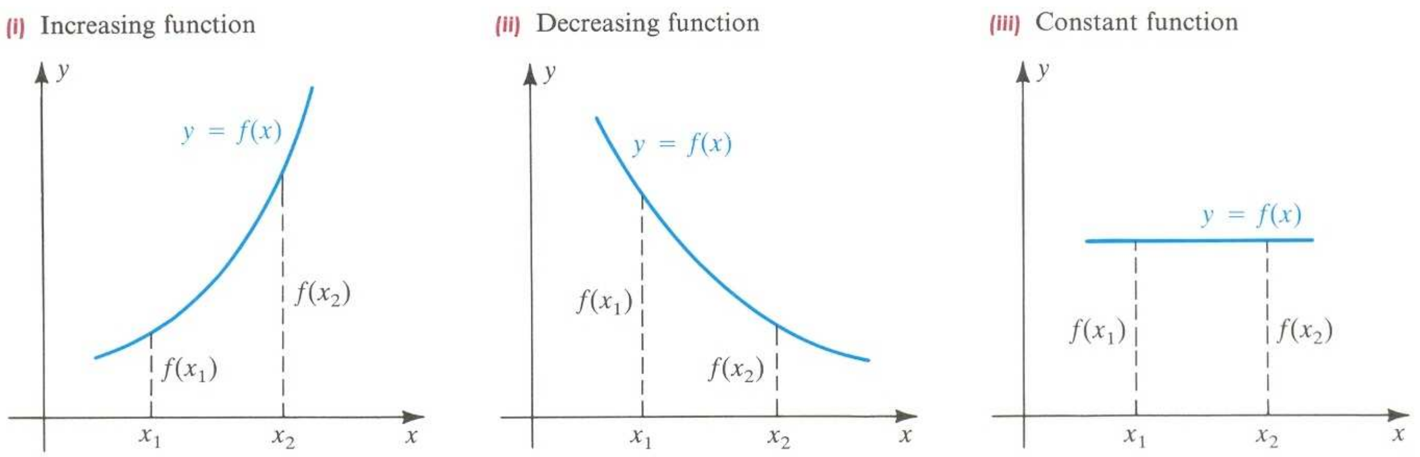
Let a function f be defined on an interval I, and let x_1, x_2 denote numbers in I.

(i) f is increasing on I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(ii) f is decreasing on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

(iii) f is constant on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

Figure 4.2



Definition (4.2) : Page (167)

Let a function f be defined on a set S of real numbers, and let c be a number in S.

(i) $f(c)$ is the maximum value of f on S if $f(x) \leq f(c)$ for every x in S.

(ii) $f(c)$ is the minimum value of f on S if $f(x) \geq f(c)$ for every x in S.

Figure 4.3

Maximum value $f(c)$

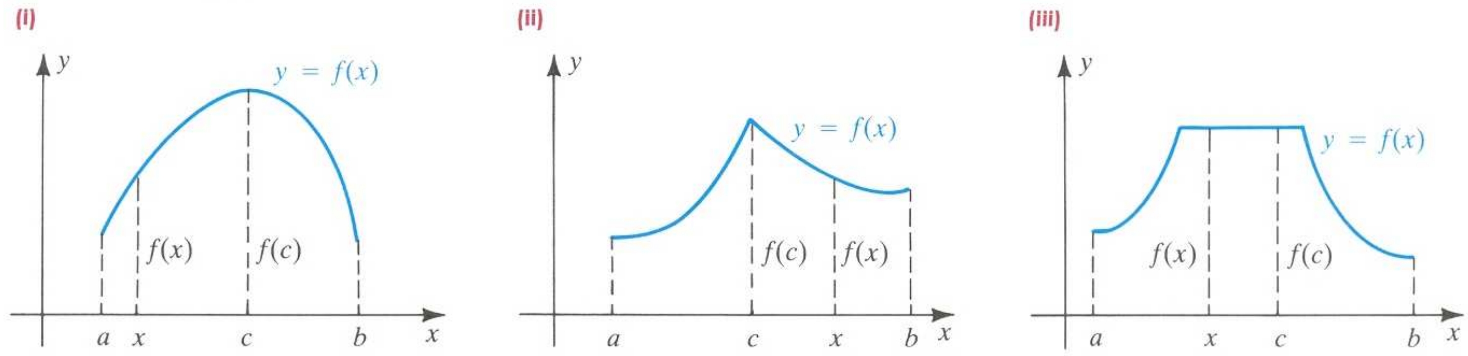
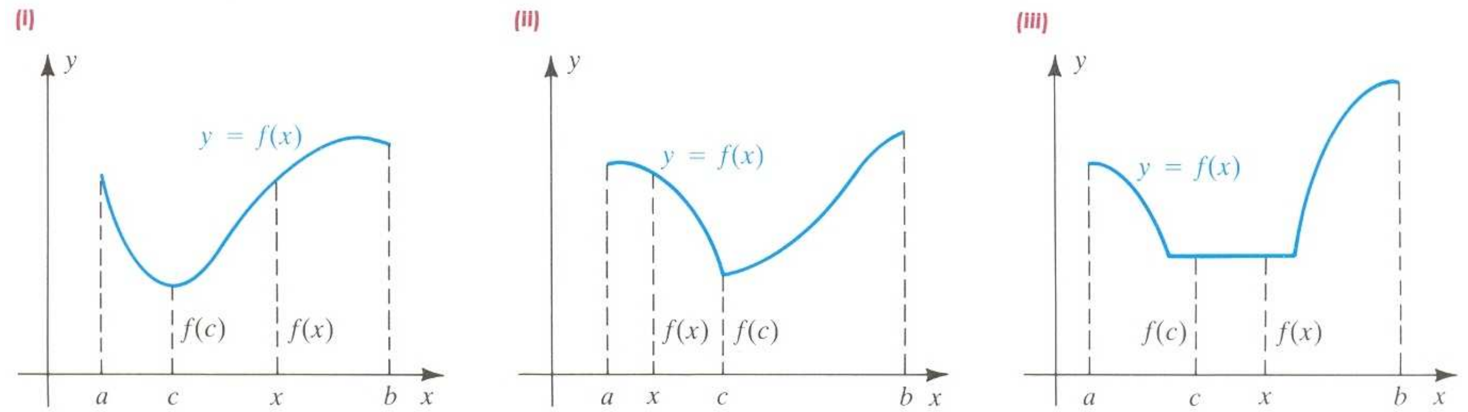


Figure 4.4

Minimum value $f\,(c)$



*** If D is the domain of f, then the maximum and minimum values of f on D, if they exist, are called the absolute maximum and absolute minimum of f.***

Example (1) : ***Page (168)***

Let $f\,(x)=4-x^2$. Find the extrema of f on the following intervals :

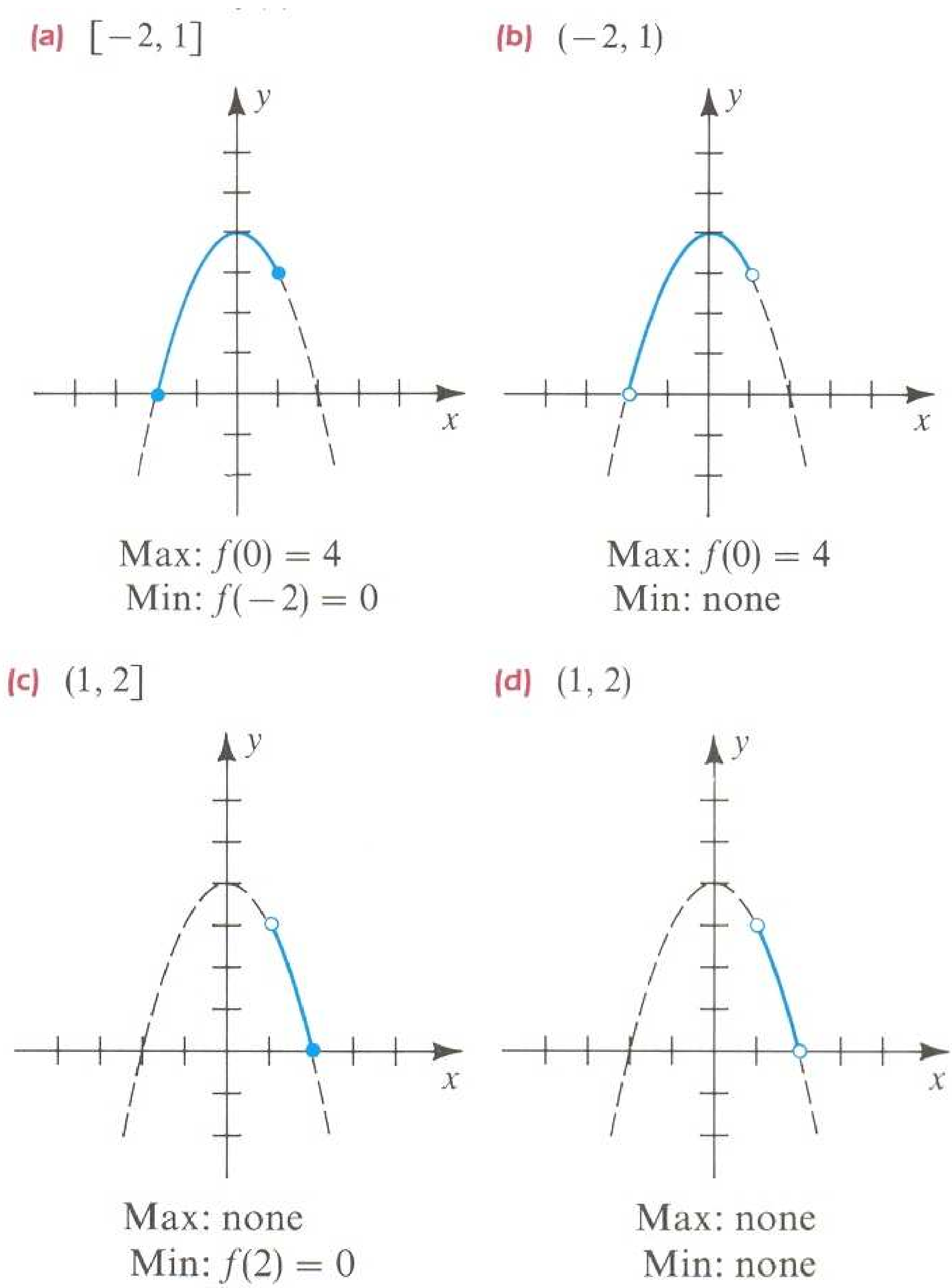
(a) $[-2,1]$. ***(b)*** $(-2,1)$. ***(c)*** $(1,2]$. ***(d)*** $(1,2)$.

Solution

*** The graph of f (a parabola) is sketched with dashes in Figure 4.5, where the solid portions correspond to the intervals (a) - (d). The extrema in each interval (denoted by Max and Min) are listed under each graph.***

Figure 4.5

$$f\,(x)=4-x^2$$



Example (2) : ***Page (169)***

Let $f(x) = \dfrac{1}{x^2}$. ***Find the extrema of*** f ***on***

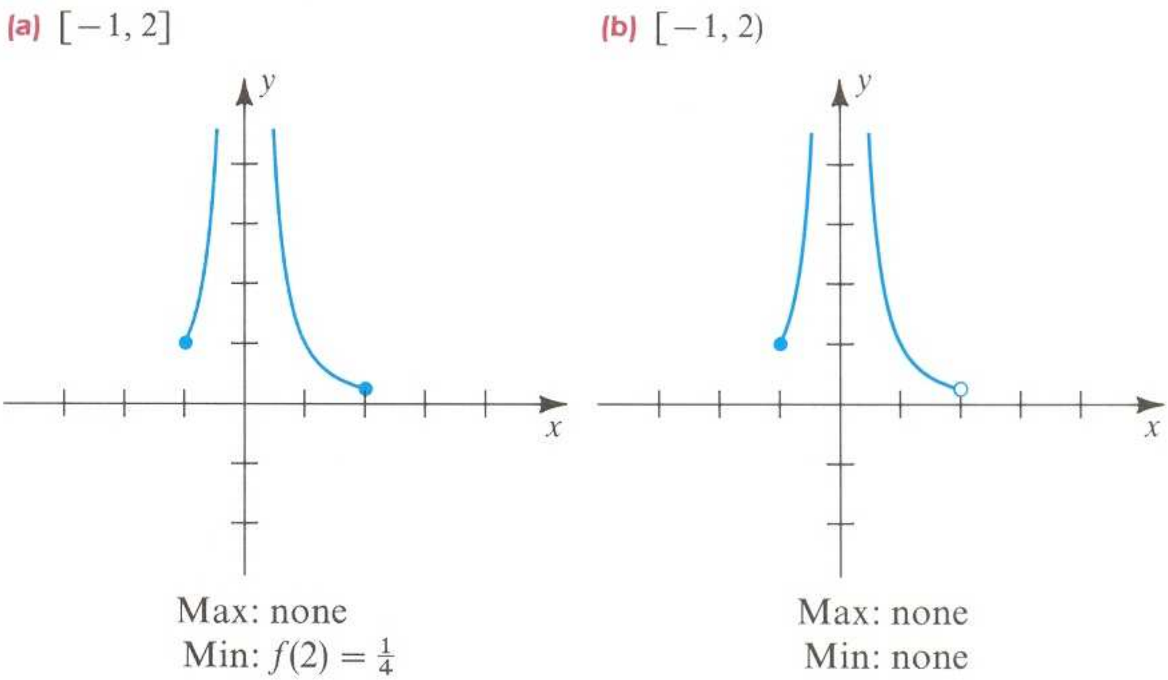
(a) $[-1, 2]$. ***(b)*** $[-1, 2)$.

Solution

** Portions of the graph of f and the extrema on the given intervals are in Figure 4.6 . Note that f is not continuous at 0 .*

Figure 4.6

$$f(x) = \frac{1}{x^2}$$



Extreme value theorem (4.3) : ***Page (169)***

If a function f is continuous on a closed interval $[a,b]$, then f takes on a minimum value and a maximum value at least once in $[a,b]$.

Definition (4.4) : ***Page (169)***

Let c be a number in the domain of a function f.

(i) $f(c)$ is a local maximum of f if there exists an open interval (a,b) containing c such that $f(x) \le f(c)$ for every x in (a,b).

(ii) $f(c)$ is a local minimum of f if there exists an open interval (a,b) containing c such that $f(x) \ge f(c)$ for every x in (a,b).

Theorem (4.5) : ***Page (170)***

If a function f has a local extremum at a number c in an open interval , then either $f'(c)=0$ or $f'(c)$ does not exist .

Corollary (4.6) : Page (170)

If $f'(c)$ exists and $f'(c)\neq 0$, then $f(c)$ is not a local extremum of the function f.

Theorem (4.7) : Page (171)

If a function f is continuous on a closed interval $[a,b]$ and has its maximum or minimum value at a number c in the open interval (a,b) , then either $f'(c)=0$ or $f'(c)$ does not exist .

Theorem (4.8) : Page (171)

A number c in the domain of a function f is a critical number of f if either $f'(c)=0$ or $f'(c)$ does not exist .

Guidelines for finding the extrema of a continuous function f on $[a,b]$ (4.9) : Page (171)

1 Find all the critical numbers of f in (a,b)
2 Calculate for each critical number c found in guideline 1.
3 Calculate the endpoint values $f(a)$ and $f(b)$.
4 The maximum and minimum values of f on $[a,b]$ are the largest and smallest function values calculated in guidelines 2 and 3 .

Example (3) : Page (172)

If $f(x)=x^3-12x$, find the maximum and minimum values of f on the closed interval $[-3,5]$ and sketch the graph of f.

Solution

$$f(x) = x^3 - 12x$$

* Using Guidelines (4.9) , we begin by finding the critical numbers of f . Differentiating yields

$$f'(x) = 3x^2 - 12 = 3\left(x^2 - 4\right) = 3(x+2)(x-2)$$

* Since the derivative exists for every x , the only critical numbers are those for which the derivative is zero - that is , -2 and 2 .

* Since f is continuous on $[-3, 5]$, it follows from our discussion that the maximum and minimum values are among the numbers $f(-2)$, $f(2)$, $f(-3)$, and $f(5)$.

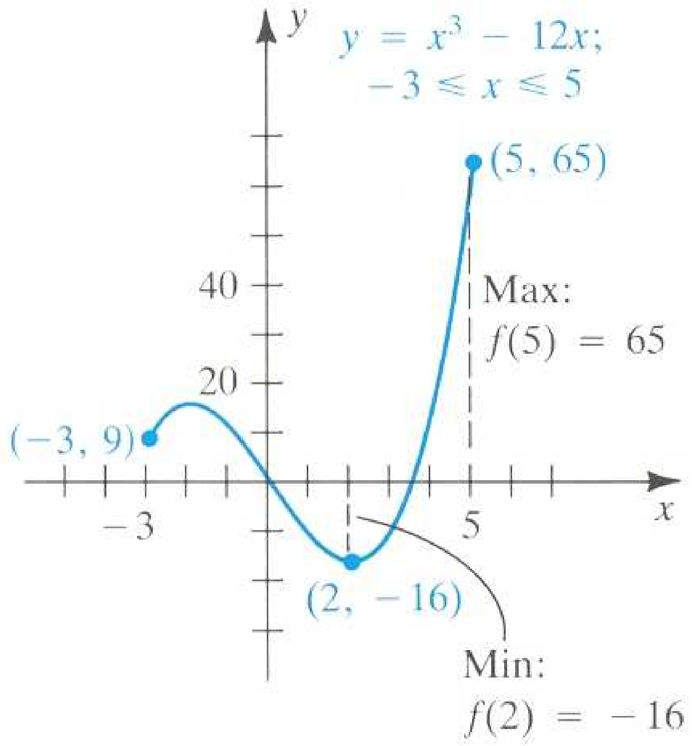
* Calculating these values (see guidelines 2 and 3) , we obtain the following table .

Values of x	Classification of x	Function value $f(x)$
-2	Critical number of f	$f(-2) = 16$
2	Critical number of f	$f(2) = -16$
-3	Endpoint of $[-3, 5]$	$f(-3) = 9$
5	Endpoint of $[-3, 5]$	$f(5) = 65$

* By guideline 4 , the minimum value of f on $[-3, 5]$ is the smallest function value $f(2) = -16$, and the maximum value is the endpoint extremum $f(5) = 65$.

* The graph of f is sketched in Figure 4.9 .

Figure 4.9



* *The tangent line is horizontal at the point corresponding to each of the critical numbers ,* -2 *and* 2 . *It will follow from our work in Section 4.4 that* $f(-2) = 16$ *is a local maximum for* f *, as indicated by the graph .*

Example (4) : ***Page (172)***

If $f(x) = (x-1)^{2/3} + 2$ *, find the maximum and minimum values of* f *on* $[0, 9]$ *, and sketch the graph of* f *.*

Solution

$$f(x) = (x-1)^{2/3} + 2$$

* *We first differentiate* $f(x)$ *:*

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

* *To find the critical numbers , we note that* $f'(x) \neq 0$ *for every* x *and that* $f'(x)$ *does not exist at* $x = 1$. *Hence* 1 *is the only critical number in* $[0, 9]$.

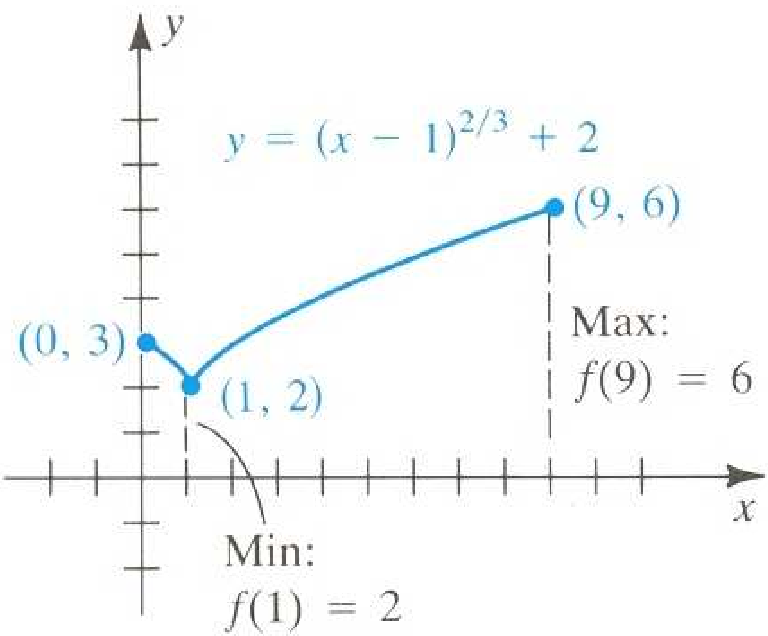
* *Let us tabulate our work :*

Values of x	Classification of x	Function value $f(x)$
1 0 9	Critical number of f Endpoint of $[0,9]$ Endpoint of $[0,9]$	$f(1)=2$ $f(0)=3$ $f(9)=6$

* Thus , by Guidelines (4.9) , f has a minimum value $f(1)=2$ and a maximum value $f(9)=6$ on the interval $[0,9]$.

* The graph of f is sketched in Figure 4.10 .

Figure 4.10



* Note that

$$\lim_{x \to 1^-} f'(x) = -\infty \quad \text{and} \quad \lim_{x \to 1^+} f'(x) = \infty .$$

* Since f is continuous at $x=1$, the graph has a cusp at $(1,2)$ by Definition (3.10) .

Example (5) : Page (173)

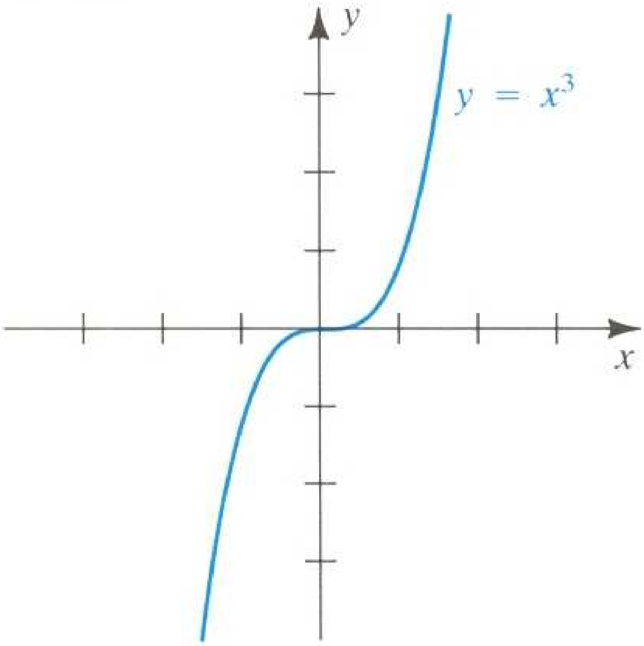
If $f(x)=x^3$, prove that f has no local extremum .

Solution

$$f(x)=x^3$$

* The of f is sketched in Figure 4.11 .

Figure 4.11



* The derivative is $f'(x)=3x^2$, which exists for every x and is zero only if $x=0$. Consequently , 0 is the only critical number .

* However , if $x<0$, then $f(x)$ is negative , and if $x>0$, then $f(x)$ is positive . Thus , $f(0)$ is neither a local maximum nor a local minimum .

* Since a local extremum must occur at a critical number (see Theorem (4.5)) , it follows that f has no local extrema . Note that the tangent line is horizontal and crosses the graph at the point $(0,0)$.

Example (6) : Page (173)

Find the critical numbers of f if $f(x)=(x+5)^2\sqrt[3]{x-4}$.

Solution

$$f(x)=(x+5)^2(x-4)^{1/3}$$

* Differentiating $f(x)$, we obtain

Remember that :

* $D_x[f(x)g(x)]=f(x)D_xg(x)+g(x)D_xf(x)$

$$f'(x)=(x+5)^2\frac{1}{3}(x-4)^{-2/3}+2(x+5)(x-4)^{1/3}.$$

* To find the critical numbers, we simplify $f'(x)$ as follows :

$$f'(x)=\frac{(x+5)^2}{3(x-4)^{2/3}}+2(x+5)(x-4)^{1/3}$$

$$=\frac{(x+5)^2+6(x+5)(x-4)}{3(x-4)^{2/3}}$$

$$=\frac{(x+5)[(x+5)+6(x-4)]}{3(x-4)^{2/3}}$$

$$=\frac{(x+5)(7x-19)}{3(x-4)^{2/3}}$$

* Hence $f'(x)=0$ if $x=-5$ or $x=\frac{19}{7}$. The derivative $f'(x)$ does not exist at $x=4$.

Thus, $\boxed{f \text{ has the three critical numbers } -5,\ \frac{19}{7},\ \text{and } 4}$

Example (7) : Page (174)

If $f(x)=2\sin x+\cos 2x$, find the critical numbers of f that are in the interval $[0,2\pi]$.

Solution

$$f(x)=2\sin x+\cos 2x$$

* Differentiating $f(x)$ gives us

$$f'(x)=2\cos x-2\sin 2x.$$

* Since $\sin 2x=2\sin x\cos x$ gives us

$$f'(x) = 2\cos x - 4\sin x \cos x .$$
$$= 2\cos x\,(1 - 2\sin x).$$

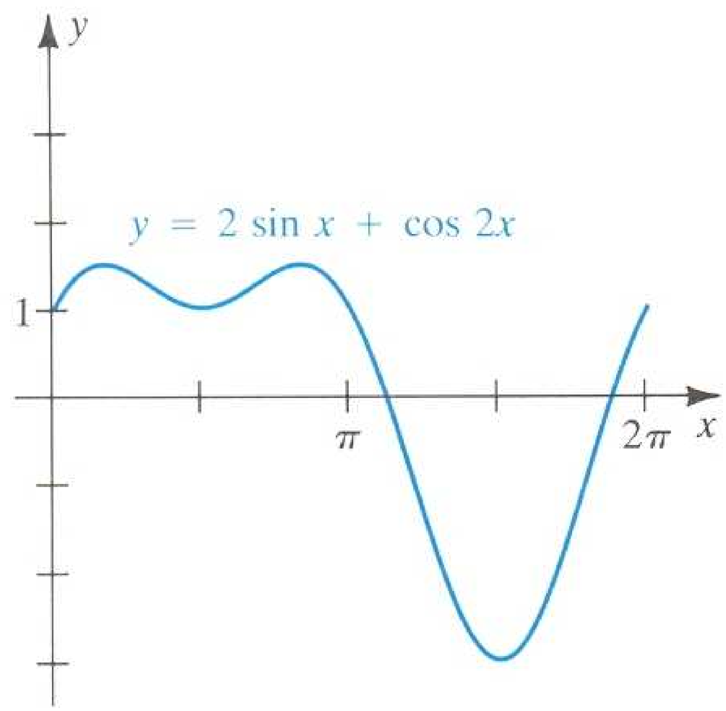
* *The derivative exists for every x, and $f'(x)=0$ if either $\sin x = \dfrac{1}{2}$ or $\cos x = 0$.*

* *Hence the critical numbers of f in the interval $[0, 2\pi]$ are*

$$\boxed{\frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{\pi}{2},\ \text{and}\ \frac{3\pi}{2}}\ .$$

* *We sketch the graph of f in Figure 4.38 .*

Figure 4.38



Home Work Exercises 4.1: 1, 3, 5, 7, 13, 17, 29, 37, 38.

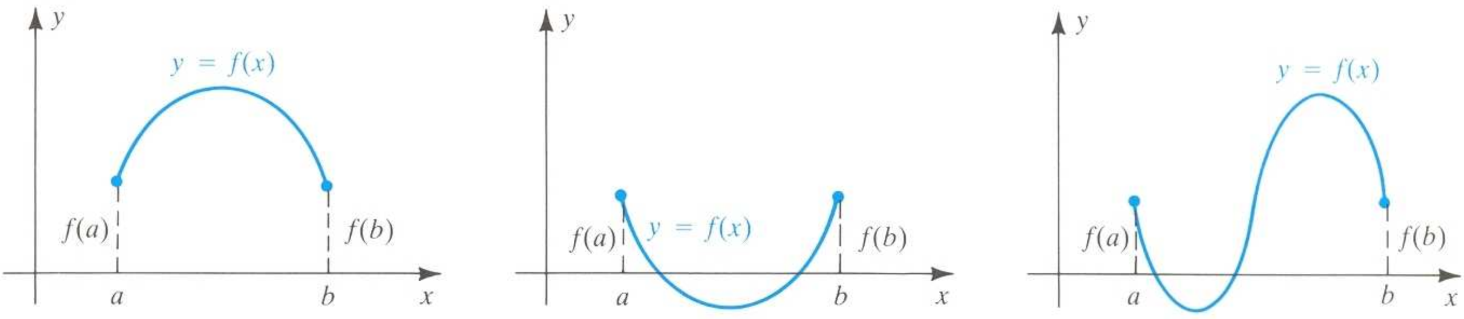
4.2 THE MEAN VALUE THEOREM : *Page (177)*

Roll's theorem (4.10) : *Page (177)*

If f is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b) and if $f(a)=f(b)$, then $f'(c)=0$ for at least one number c in

(a, b).

Figure 4.12



Corollary (4.11) : **Page (177)**

If f is continuous on a closed interval $[a, b]$ and $f(a) = f(b)$, then f has at least one critical number in the open interval (a, b).

Example (1) : **Page (178)**

Let $f(x) = 4x^2 - 20x + 29$. Show that f satisfies the hypotheses of Roll's theorem on the interval $[1, 4]$, and find all real numbers c in the open interval $(1, 4)$ such that $f'(c) = 0$. Illustrate the results graphically.

Solution

$$f(x) = 4x^2 - 20x + 29$$

* *Since f is a polynomial function, it is continuous and differentiable for every x. In particular, it is continuous on $[1, 4]$ and differentiable on $(1, 4)$.*

* *Moreover,*

$$f(1) = 4 - 20 + 29 = 13$$

$$f(4) = 4 - 80 + 29 = 13$$

and hence $f(1) = f(4)$. Thus, f satisfies the hypotheses of Rolle's theorem on $[1, 4]$.

* *Differentiating $f(x)$, we have*

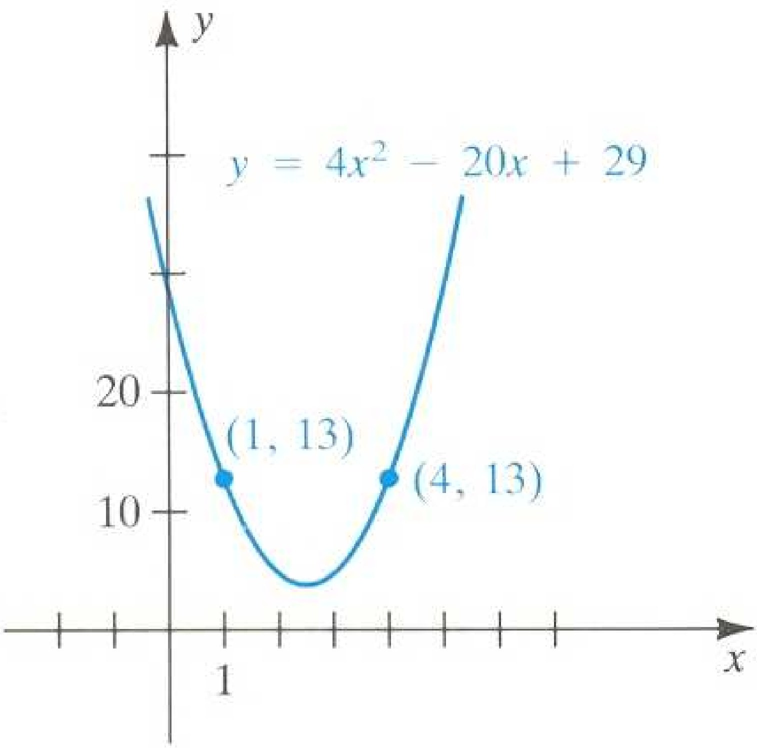
$$f'(x) = 8x - 20 .$$

* Setting $f'(x) = 0$ gives us $8x = 20$, or $x = \frac{5}{2}$. Hence

$$\boxed{f'\left(\frac{5}{2}\right) = 0 \quad \text{and} \quad 1 < \frac{5}{2} < 4} .$$

* The graph of f (a parabola) is sketched in Figure 4.13 . Since $f'\left(\frac{5}{2}\right) = 0$, the tangent line is horizontal at the vertex $\left(\frac{5}{2}, 4\right)$.

Figure 4.13



Mean value theorem (4.12) : Page (179)

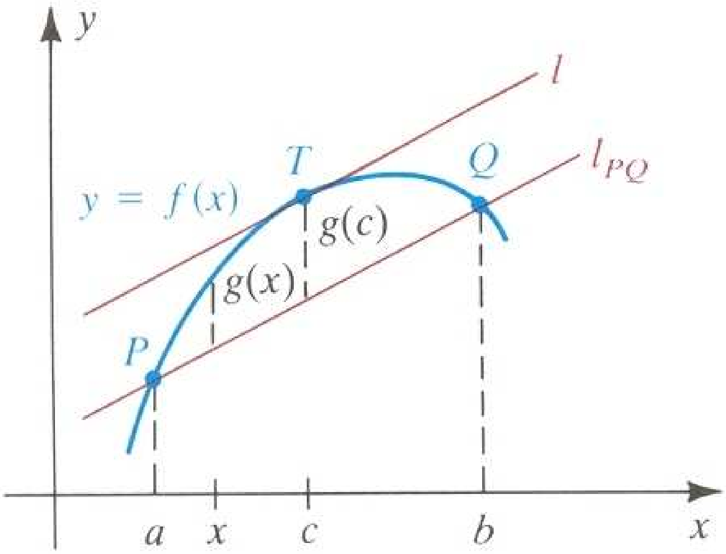
If f is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b) , then there exists a number c in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a},$$

or , equivalently ,

$$f(b)-f(a)=f'(c)(b-a).$$

Figure 4.15



Example (2) : ***Page (180)***

Let $f(x)=\frac{1}{4}x^2+1.$ *Show that f satisfies the hypotheses of the mean value theorem on the interval* $[-1,4]$*, and find a numbers c in* $(-1,4)$ *that satisfies the conclusion of the theorem . Illustrate the results graphically .*

Solution

$$f(x)=\frac{1}{4}x^2+1$$

* *The quadratic function f is continuous on* $[-1,4]$ *and differentiable on* $(-1,4)$ *;*
hence , by the mean value theorem, there is a number c in $(-1,4)$ *such that*

$$\frac{f(4)-f(-1)}{4-(-1)}=f'(c)$$

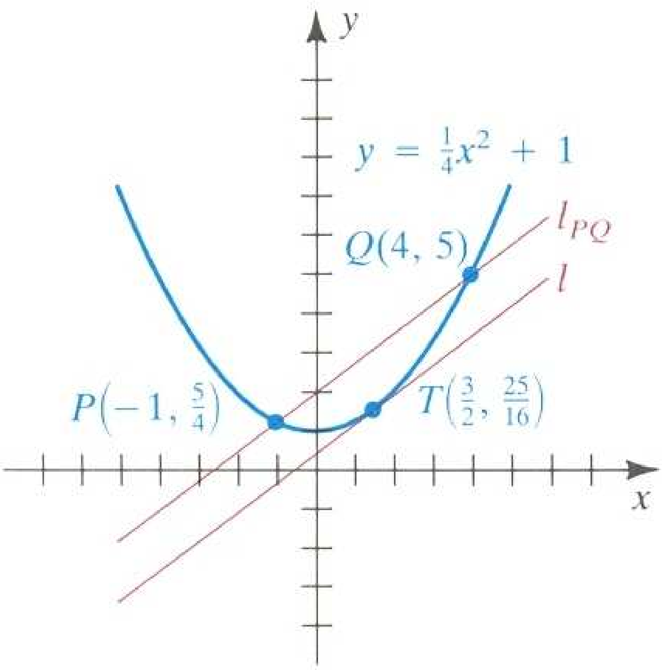
* Using $f(4)=5$, $f(-1)=\frac{5}{4}$, and $f'(x)=\frac{1}{2}x$ gives us

$$\frac{5-\frac{5}{4}}{5}=\frac{1}{2}c \ , \quad \text{or} \quad \frac{3}{4}=\frac{1}{2}c$$

* Thus , $\boxed{c=\frac{3}{2}}$.

* The graph of f (a parabola) is sketched in Figure 4.16 . The points $P\left(-1,\frac{5}{4}\right)$ and $Q(4,5)$ correspond to the endpoints of the interval $[-1,4]$. The point $T\left(\frac{3}{2},\frac{25}{16}\right)$ is obtained by using $c=\frac{3}{2}$ and is the point at which the tangent line l is parallel to the secant line l_{PQ} .

Figure 4.16



__*Example (3) :*__ *Page (180)*

Let $f(x)=x^3-8x-5$. Show that f satisfies the hypotheses of the mean value theorem on the interval $[1,4]$, and find a numbers c in the open interval $(1,4)$ that satisfies the conclusion of the theorem .

Solution

$$f(x)=\frac{1}{4}x^2+1$$

* Since f is a polynomial function , it is continuous and differentiable for all real numbers . In particular , it is continuous on $[1,4]$ and differentiable on the open interval $(1,4)$. By the mean value theorem, there is a number c in $(1,4)$ such that

$$\frac{f(4)-f(1)}{4-(-1)}=f'(c)$$

* Using $f(4)=27$, $f(-1)=-12$, and $f'(x)=3x^2-8$ gives us

$$\frac{27-(-12)}{3}=3c^2-8 \quad , \quad \text{or} \quad 13=3c^2-8$$

* Thus , $c^2=7$, or $c=\sqrt{7}$. Since $-\sqrt{7}$ is not in the interval $(1,4)$, the desired number is $\boxed{c=\sqrt{7}}$.

Home Work Exercises 4.2: 3 , 7, 11, 19, 25.

4.3 THE FIRST DERIVATIVE TEST : Page (183)

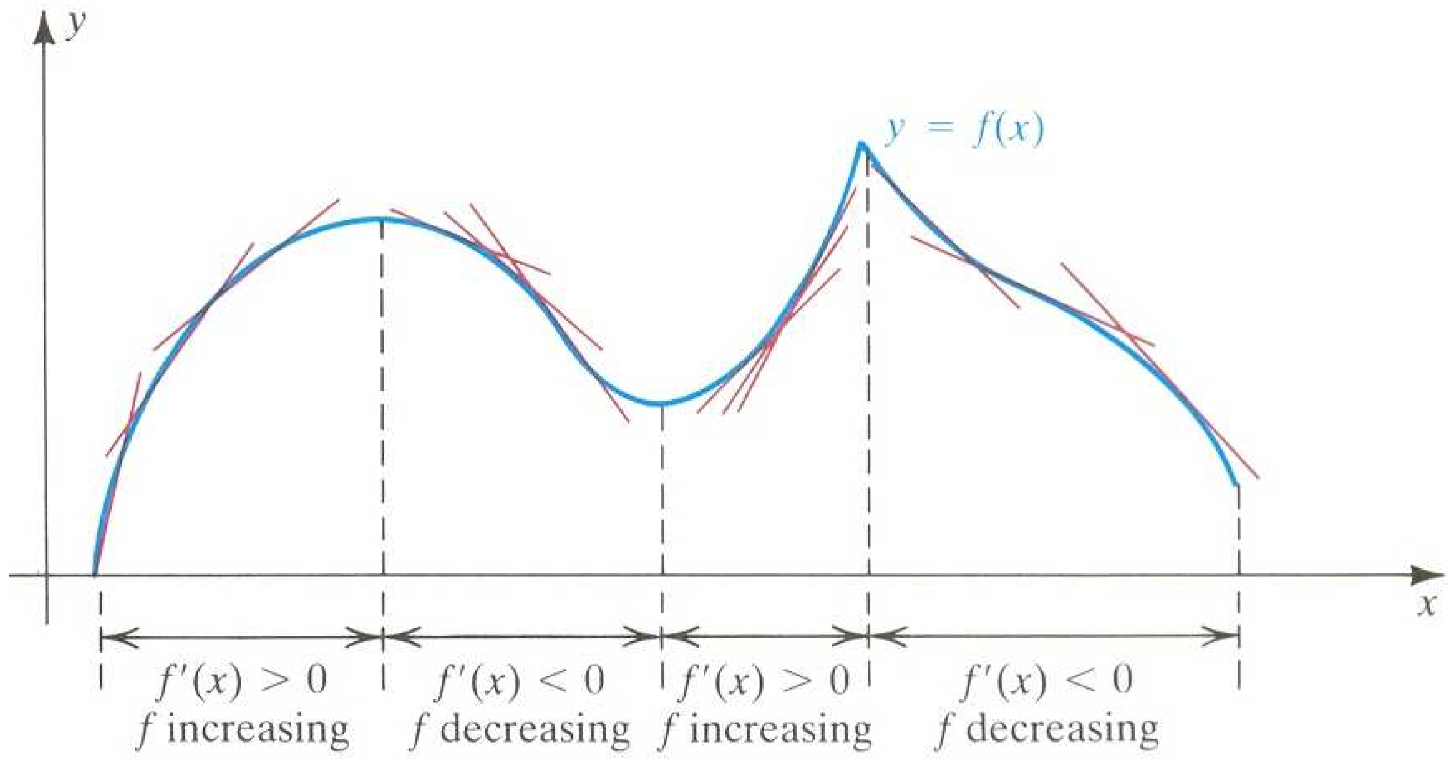
Theorem (4.13) : Page (183)

Let a function f be defined on $[a,b]$ and differentiable on (a,b).

(i) If $f'(x)>0$ for every x in (a,b), then f is increasing on $[a,b]$.

(ii) If $f'(x)<0$ for every x in (a,b), then f is decreasing on $[a,b]$.

Figure 4.18



Example (1) : ***Page (184)***

If $f(x) = x^3 + x^2 - 5x - 5$,

(a) Find the interval on which f is increasing and the intervals on which f is decreasing .

(b) Sketch the graph of f .

Solution

$$f(x) = x^3 + x^2 - 5x - 5$$

(a) First we differentiate :

$$f'(x) = 3x^2 + 2x - 5 = (3x+5)(x-1)$$

**** By Theorem (4.13) , it is sufficient to find the intervals in which $f'(x) > 0$ and those in which $f'(x) < 0$. The factored form of $f'(x)$ and the critical numbers $-\frac{5}{3}$ and 1 suggest the open intervals $\left(-\infty, -\frac{5}{3}\right)$, $\left(-\frac{5}{3}, 1\right)$, and $(1, \infty)$.***

**** On each of these intervals f' is continuous and has no zeros , and therefore $f'(x)$ has the same sign throughout the interval . This sign can be determined by choosing a suitable test value for the interval .***

* The following table displays our work . The values of k were chosen for convenience . we chose $k = -2$ in interval $\left(-\infty, -\frac{5}{3}\right)$, but any number , such as 0 or 2 , could have been used .

Interval	$\left(-\infty, -\frac{5}{3}\right)$	$\left(-\frac{5}{3}, 1\right)$	$(1, \infty)$
k	-2	0	2
Test value $f'(k)$	$f'(-2) = 3 > 0$	$f'(0) = -5 < 0$	$f'(2) = 11 > 0$
Sign of $f'(x)$	$+$	$-$	$+$
Conclusion	f is increasing on $\left(-\infty, -\frac{5}{3}\right]$	f is decreasing on $\left[-\frac{5}{3}, 1\right]$	f is increasing on $[1, \infty)$

(b) As an aid to sketching the graph of f , we shall find the x-intercepts by solving the equation $f(x) = 0$. Since

$$f(x) = x^3 + x^2 - 5x - 5$$

$$= x^2(x+1) - 5(x+1)$$

$$= (x^2 - 5)(x+1),$$

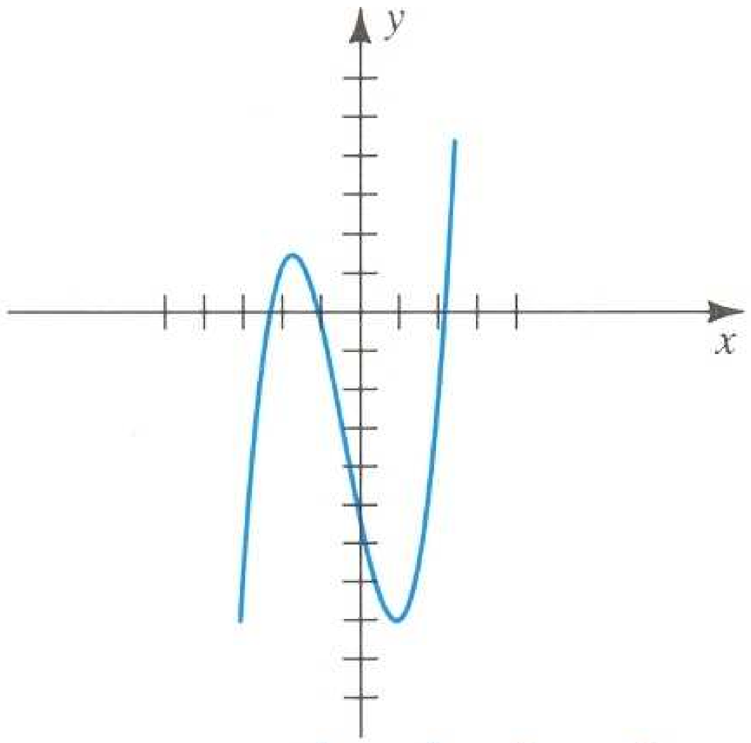
we see that the x-intercepts are $\sqrt{5}$, $-\sqrt{5}$, and -1 .

* The y-intercept is $f(0) = -5$.

* The points corresponding to the critical numbers are $\left(-\frac{5}{3}, \frac{40}{27}\right)$ and $(1, -8)$.

* Plotting these six points and using the information in the table gives us the sketch in Figure 4.19 .

Figure 4.19



$y = x^3 + x^2 - 5x - 5$

Theorem (4.14) : *Page (185)*

Let c be a critical number for f , and suppose f is continuous at c and differentiable on an open interval I containing c , except possibly at c itself .

(i) If f' changes from positive to negative at c , then $f(c)$ is a local maximum of f .

(ii) If f' changes from negative to positive at c , then $f(c)$ is a local maximum of f .

(iii) If $f'(x) > 0$ or if $f'(x) < 0$ for every x in I except $x = c$, then $f(c)$ is not a local extremum of f .

Figure 4.20

local maximum $f(c)$

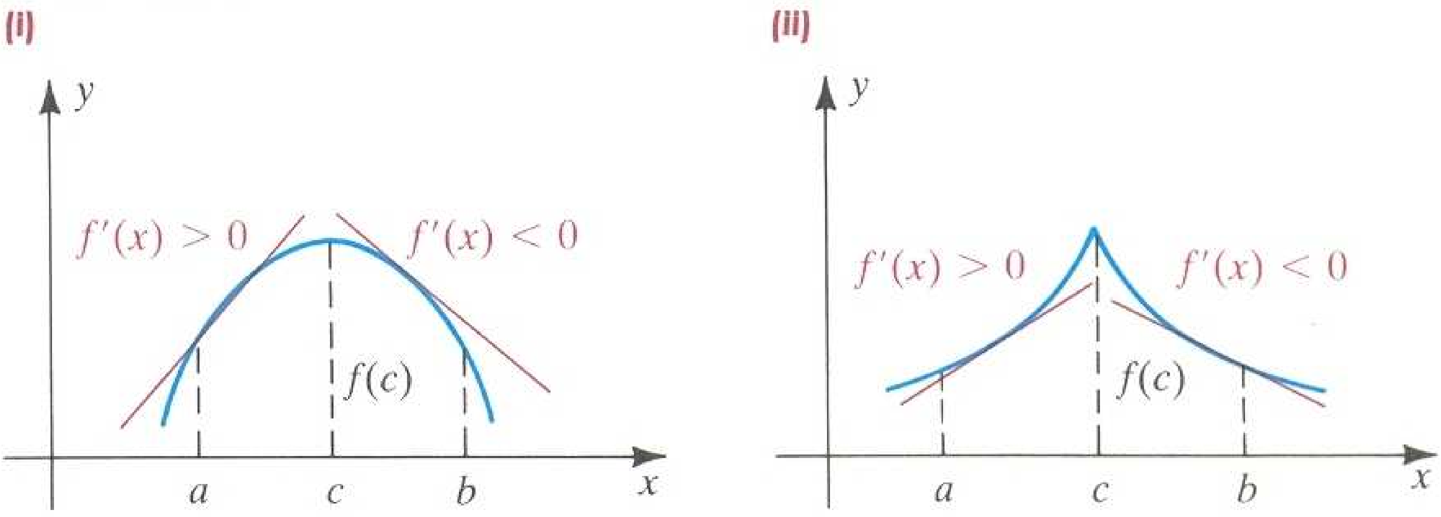


Figure 4.21

local maximum $\boldsymbol{f(c)}$

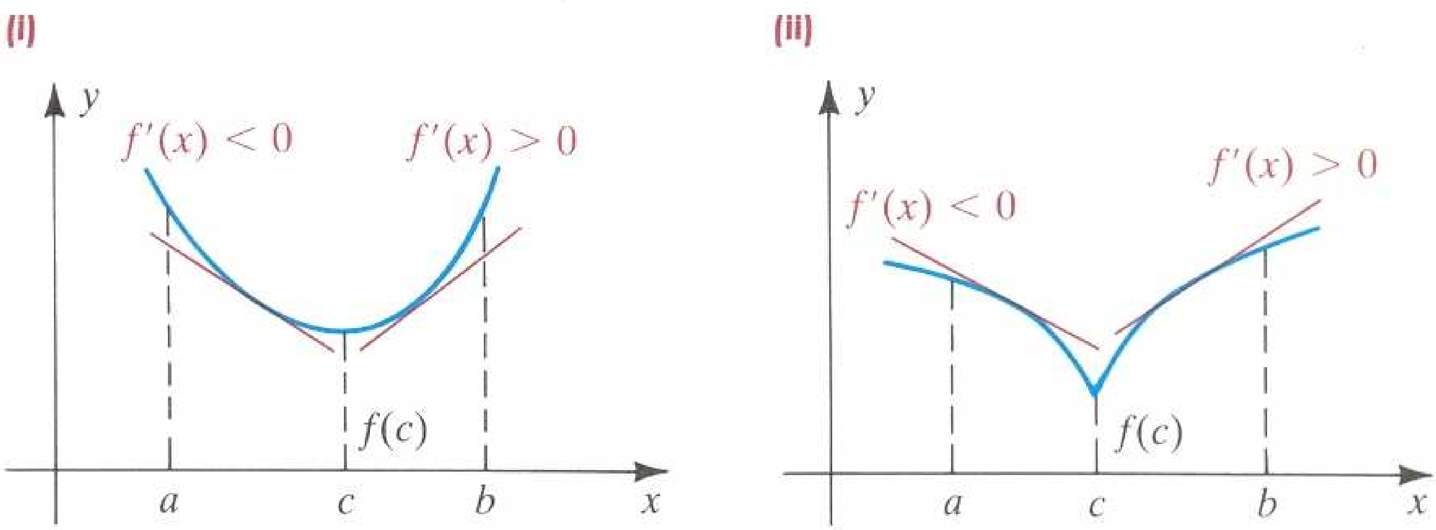
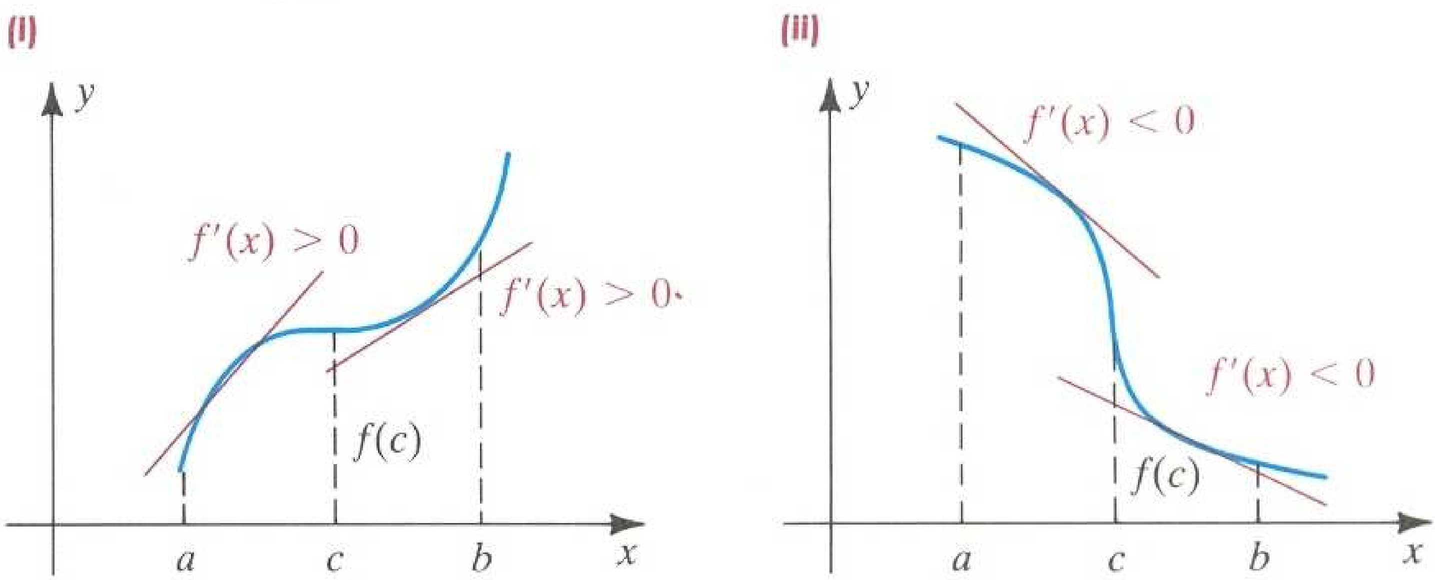


Figure 4.22

$\boldsymbol{f(c)}$ ***is not a local extremum***



Example (2) : Page (186)

If $f(x) = x^3 + x^2 - 5x - 5$, find the local extrema of f.

Solution

$$f(x) = x^3 + x^2 - 5x - 5$$

* This is the function considered in Example (1). The critical numbers are $-\frac{5}{3}$ and 1.

* We see from the table in Example (1) that the sign of $f'(x)$ changes from positive to negative as x increases through $-\frac{5}{3}$.

* Hence, by the first derivative test, f has a local maximum at $-\frac{5}{3}$. This maximum value is $f\left(-\frac{5}{3}\right) = \frac{40}{27}$ (see Figure 4.19).

* A local minimum occurs at 1, since the sign of $f'(x)$ changes from negative to positive as x increases through 1. This minimum value is $f(1) = -8$.

Example (3) : Page (186)

If $f(x) = x^{1/3}(8 - x)$, find the local extrema of f, and sketch the graph of f.

Solution

$$f(x) = x^{1/3}(8 - x)$$

* By the product rule,

Remember that :

$$* \quad D_x[f(x)g(x)] = f(x)D_x g(x) + g(x)D_x f(x)$$

$$f'(x) = x^{1/3}(-1) + (8 - x)\frac{1}{3}x^{-2/3}$$

$$= \frac{-3x + (8-x)}{3x^{2/3}} = \frac{4(2-x)}{3x^{2/3}}.$$

* *Hence the critical numbers of f are 0 and 2. As in Example (1), this suggests that we consider the sign of $f'(x)$ in each of the intervals $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$.*

* *Since f' is continuous and has no zeros on each interval, we may determine the sign of $f'(x)$ by using a suitable test value $f'(k)$. It is unnecessary to actually evaluate $f'(k)$; all we need to know is its sign. Thus, if we choose $k = 3$ in $(2, \infty)$, then*

$$f'(x) = \frac{4(2-3)}{3\left(3^{2/3}\right)}$$

and we can tell, without evaluating, that the numerator is negative and the denominator is positive. Hence $f'(3) < 0$, as shown in the following table.

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
k	-1	1	3
Test value $f'(k)$	$f'(-1) > 0$	$f'(1) > 0$	$f'(3) < 0$
Sign of $f'(x)$	$+$	$+$	$-$
Conclusion	*f is increasing on* $(-\infty, 0]$	*f is increasing on* $[0, 2]$	*f is decreasing on* $[2, \infty)$

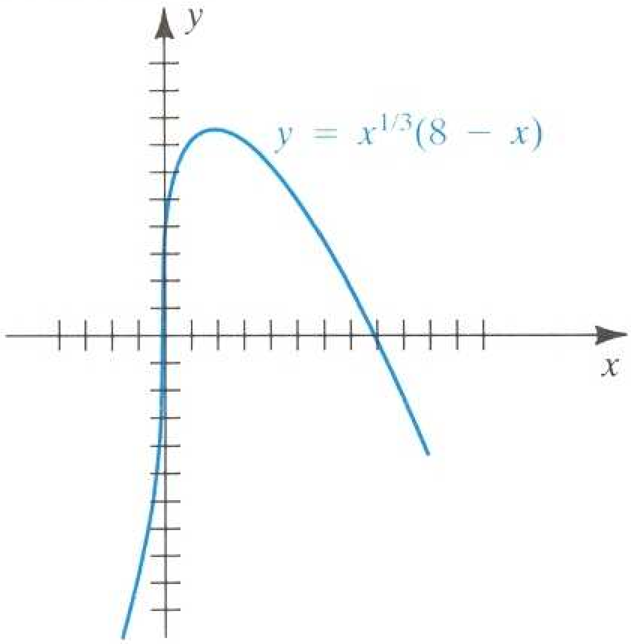
* *By the first derivative test, f has a local maximum at 2, since f' changes from positive to negative at 2. Thus, we have*

$$\textit{Local max}: \quad f(2) = 2^{1/3}(8-2) = 6\sqrt[3]{2} \approx 7.6 .$$

* *The function does not have an extremum at 0, since f' does not change sign at 0.*

* To sketch the graph , we first plot points corresponding to the critical numbers . From $f(x) = x^{1/3}(8-x)$ we see that the x-intercepts of the graph are 0 and 8 . The graph is sketched in Figure 4.23 .

Figure 4.23



* Note that

$$\lim_{x \to 0} f'(x) = \infty .$$

* Since f is continuous at $x = 0$, the graph has a vertical tangent line at $(0,0)$ by Definition (3.9) .

Example (4) : Page (187)

If $f(x) = x^{2/3}(x^2 - 8)$, find the local extrema , and sketch the graph of f .

Solution

$$f(x) = x^{2/3}(x^2 - 8)$$

* Applying the product rule , we obtain

Remember that :

$$* \; D_x[f(x)g(x)] = f(x)D_x g(x) + g(x)D_x f(x)$$

$$f'(x) = x^{2/3}(2x) + (x^2 - 8)\left(\frac{2}{3}x^{-1/3}\right)$$

$$= \frac{6x^2 + 2(x^2 - 8)}{3x^{1/3}} = \frac{8(x^2 - 2)}{3x^{1/3}} .$$

* The critical numbers are the solutions of $x^2 - 2$ and $x^{1/3}$ -that is , $-\sqrt{2}$, 0 , and $\sqrt{2}$. This suggests that we find the sign of $f'(x)$ in each of the intervals $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, 0)$, $(0, \sqrt{2})$, and $(\sqrt{2}, \infty)$.

* Arrange our work in tabular form , we obtain the following .

Interval	$(-\infty, -\sqrt{2})$	$(-\sqrt{2}, 0)$	$(0, \sqrt{2})$	$(\sqrt{2}, \infty)$
k	-2	-1	1	2
Test value $f'(k)$	$f'(-2) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(2) > 0$
Sign of $f'(x)$	$-$	$+$	$-$	$+$
Conclusion	f is decreasing on $(-\infty, -\sqrt{2}]$	f is increasing on $[-\sqrt{2}, 0]$	f is decreasing on $[0, \sqrt{2}]$	f is increasing on $[\sqrt{2}, \infty)$

* By the first derivative test , f has a local minimum at $-\sqrt{2}$ and $\sqrt{2}$ and a local maximum at 0 . The corresponding function values give us the following results .

Local max : $f(0) = 0$.

Local min : $f(\sqrt{2}) = f(-\sqrt{2}) = -6\sqrt[3]{2} \approx -7.6$.

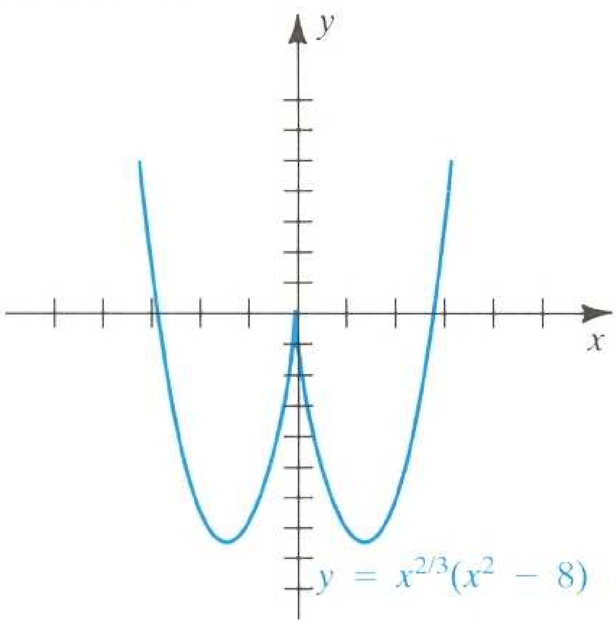
* Note that $f'(0)$ does not exist . Since

$$\lim_{x \to 0^-} f'(x) = \infty \quad \text{and} \quad \lim_{x \to 0^+} f'(x) = -\infty$$

and since f is continuous at $x = 0$, the graph has a cusp at $(0, 0)$, by Definition (3.10) .

* The graph of f is sketched in Figure 4.24 .

Figure 4.24



Example (5) : ***Page (188)***

If $f(x) = x^{2/3}(x^2 - 8)$***, find the maximum and minimum values of f on each of the following intervals :***

(a) $\left[-1, \frac{1}{2}\right]$. ***(b)*** $[-1, 3]$. ***(c)*** $[-3, -2]$.

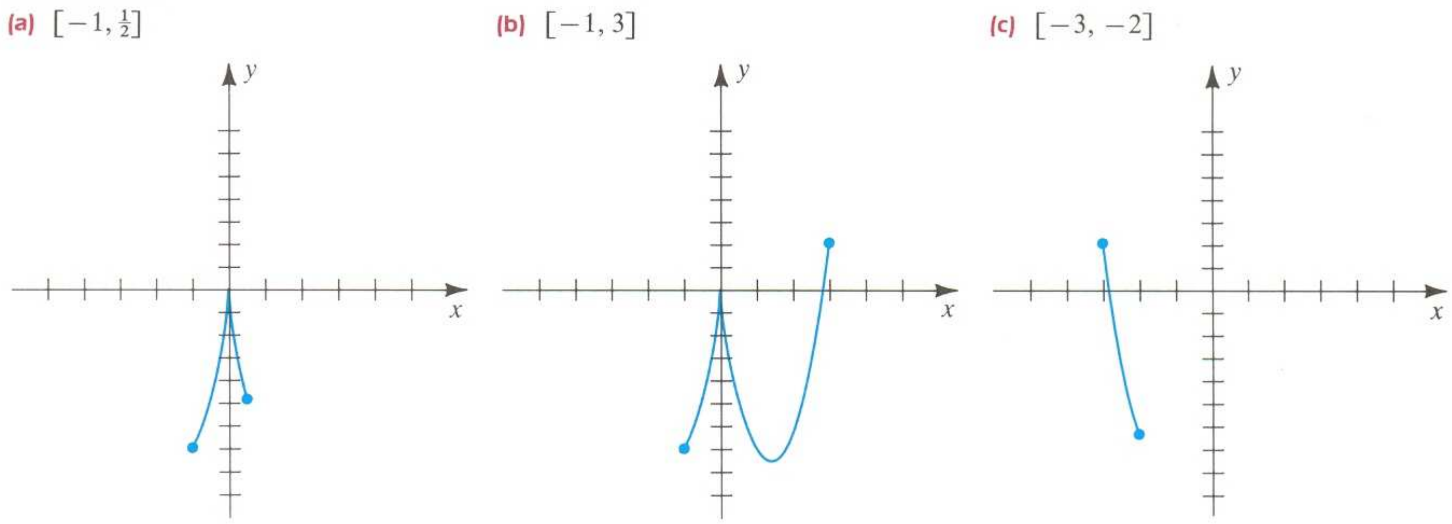
Solution

$$f(x) = x^{2/3}(x^2 - 8)$$

** The graph in Figure 4.24 indicates the local extrema and the intervals on which f is increasing or decreasing .*

** Figure 4.25 illustrates the part of the graph of f that corresponds to each of the intervals (a) , (b) , and (c) .*

Figure 4.25



* *Referring to these sketches , we obtain the following table (check each entry)*

Interval	*Minimum value*	*Maximum value*
$\left[-1,\frac{1}{2}\right]$	$f(-1)=-7$	$f(0)=0$
$[-1,3]$	$f(\sqrt{2})=-6\sqrt[3]{2}$	$f(3)=\sqrt[3]{9}$
$[-3,-2]$	$f(-2)=-4\sqrt[3]{4}$	$f(-3)=\sqrt[3]{9}$

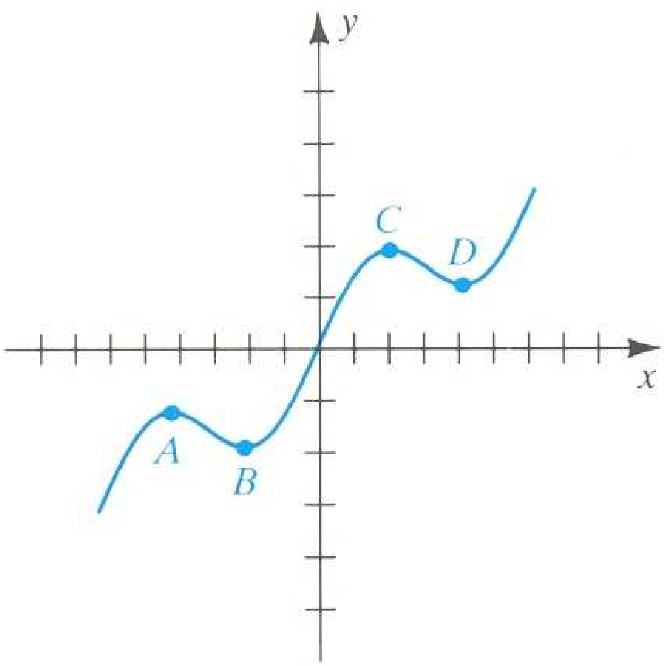
* *Note that on some intervals the maximum or minimum value of f is also a local extremum ; on other intervals this is not the case .*

Example (6) : *Page (189)*

The graph of $y=\frac{1}{2}x+\sin x$ for $-2\pi \le x \le 2\pi$ *is sketched in Figure 4.26 . Determine the coordinates of the points A , B , C , and D , which correspond to local extrema .*

Solution

Figure 4.26



* *Letting* $f(x)=\frac{1}{2}x+\sin x$ *and differentiating , we obtain*

$$f'(x)=\frac{1}{2}+\cos x$$

* The local extrema occur if $f'(x)=0$ - that is , if

$$\frac{1}{2}+\cos x=0\ ,\quad \text{or}\quad \cos x=-\frac{1}{2}$$

* The solutions of the last equation in $[0,2\pi]$ are $2\pi/3$ and $4\pi/3$. These are the x-coordinates of C and D . By symmetry (with respect to the origin) , the x-coordinates of A and B are $-4\pi/3$ and $-2\pi/3$.

* We could now apply the first derivative test ; however , it is evident from the graph that points A and C correspond to local maxima and points B and D correspond to local minima . The following table lists the coordinates of these points .

Point	x-coordinate	y-coordinate
A	$-\frac{4\pi}{3}\approx-4.2$	$-\frac{2\pi}{3}+\frac{\sqrt{3}}{2}\approx-1.2$
B	$-\frac{2\pi}{3}\approx-2.1$	$-\frac{\pi}{3}+\left(-\frac{\sqrt{3}}{2}\right)\approx-1.9$
C	$\frac{2\pi}{3}\approx2.1$	$\frac{\pi}{3}+\frac{\sqrt{3}}{2}\approx1.9$
D	$\frac{4\pi}{3}\approx4.2$	$-\frac{2\pi}{3}+\left(-\frac{\sqrt{3}}{2}\right)\approx1.2$

Exercises 4.3: 5, 9, 17, 23, 27.

4.4 CONCAVITY AND THE SECOND DERIVATIVE TEST :

Page (191)

Definition (4.15) : Page (191)

Let f be differentiable on an open interval I . The graph of f is

(i) concave upward on I if f' is increasing on I .

(ii) concave downward on I if f' is decreasing on I .

Figure 4.27

Concave upward grap

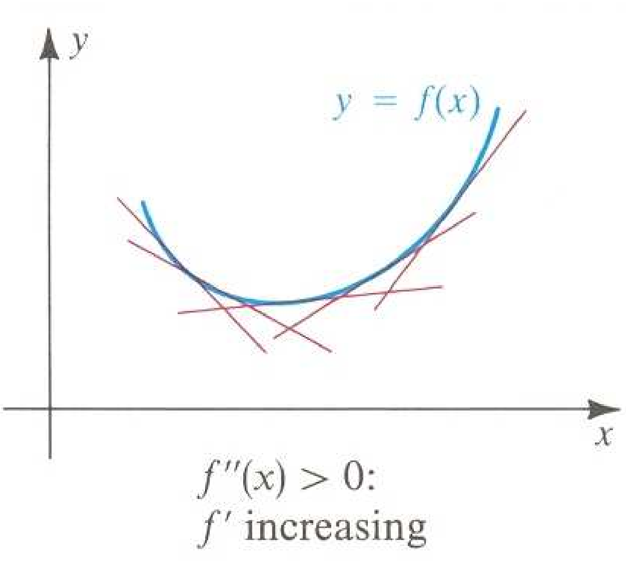
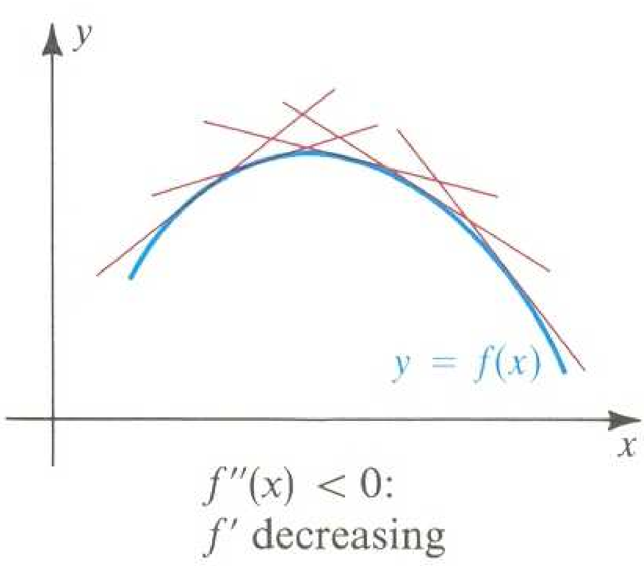


Figure 4.28

Concave downward graph



Test for concavity (4.16) : ***Page (192)***

If the second derivative f'' of f exists on an open interval I , then the graph of f is

(i) concave upward on I if $f''(x) > 0$ on I .

(ii) concave downward on I if $f''(x) < 0$ on I .

Example (1) : ***Page (192)***

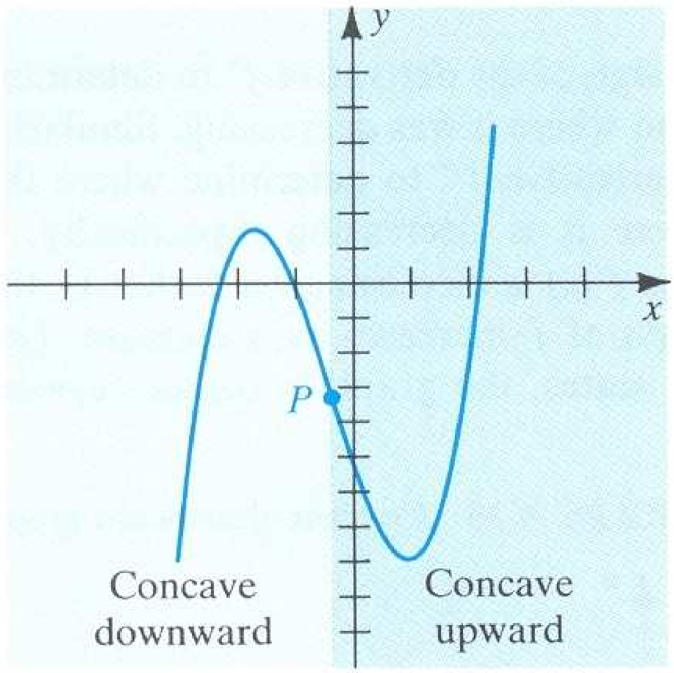
If $f(x) = x^3 + x^2 - 5x - 5$, determine intervals on which the graph of f is concave upward or is concave downward , and illustrate the results graphically .

Solution

$$f(x) = x^3 + x^2 - 5x - 5$$

**** The function f was considered in Example (1) and (2) of the preceding section and is resketched in Figure 4.29 .***

Figure 4.29



$y = x^3 + x^2 - 5x - 5$

* *Since*

$$f'(x) = 3x^2 + 2x - 5,$$

$$f''(x) = 6x + 2 = 2(3x + 1).$$

* *Hence* $f''(x) < 0$ *if* $3x + 1 < 0$ *-that is , if* $x < -\frac{1}{3}$. *Similarly ,*

$f''(x) > 0$ *if* $x > -\frac{1}{3}$.

* *Applying the test for concavity (4.16) gives us the following .*

Interval	$\left(-\infty, -\frac{1}{3}\right)$	$\left(-\frac{1}{3}, \infty\right)$
Sign of $f''(x)$	$-$	$+$
Concavity	downward	upward

* *These facts are illustrated in Figure 4.29 , where* p *is the point with x-coordinate* $-\frac{1}{3}$.

Example (2) : *Page (192)*

If $f\left(x\right)=\sin x$ *, determine where the graph of* f *is concave upward and where it is concave downward , and illustrate the results graphically .*

Solution

$$f\left(x\right)=\sin x$$

** Differentiating* $f\left(x\right)$ *twice , we obtain*

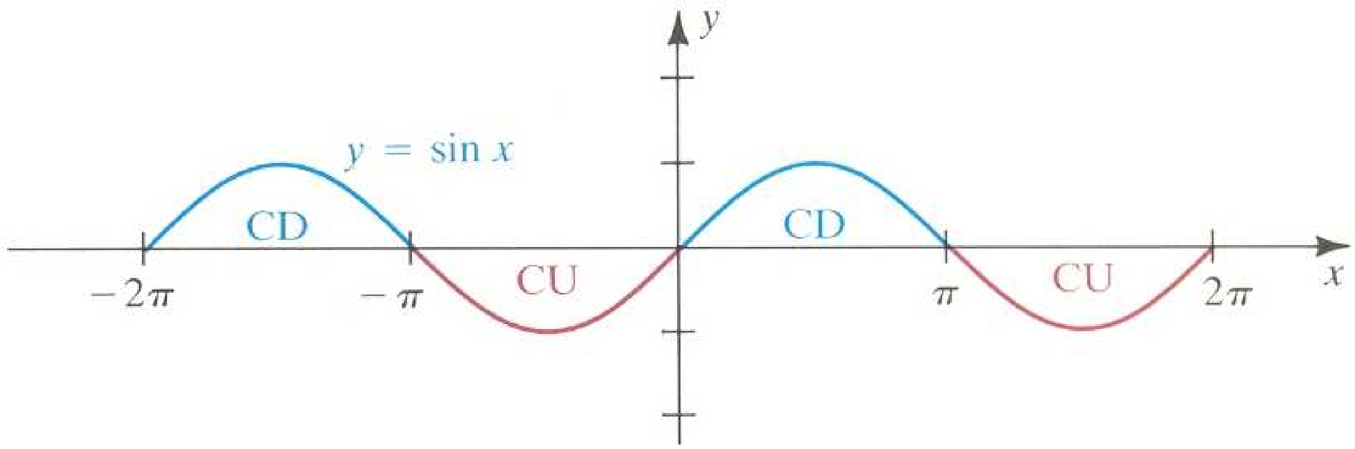
$$f'\left(x\right)=\cos x \quad , \quad f''\left(x\right)=-\sin x$$

** Since* $f''\left(x\right)=-f'\left(x\right)$ *, we see that* $f''\left(x\right)<0$ *whenever* $f\left(x\right)>0$ *; hence , by the test for concavity (4.16) , the graph is concave downward whenever it lies above the x-axis .*

** Similarly ,* $f''\left(x\right)>0$ *whenever* $f\left(x\right)<0$ *, so the graph is concave upward whenever it lies below the x-axis .*

** These facts are partially illustrated in Figure 4.30 , in which the abbreviations CD and CU are used for concave downward and concave upward , respectively .*

Figure 4.30



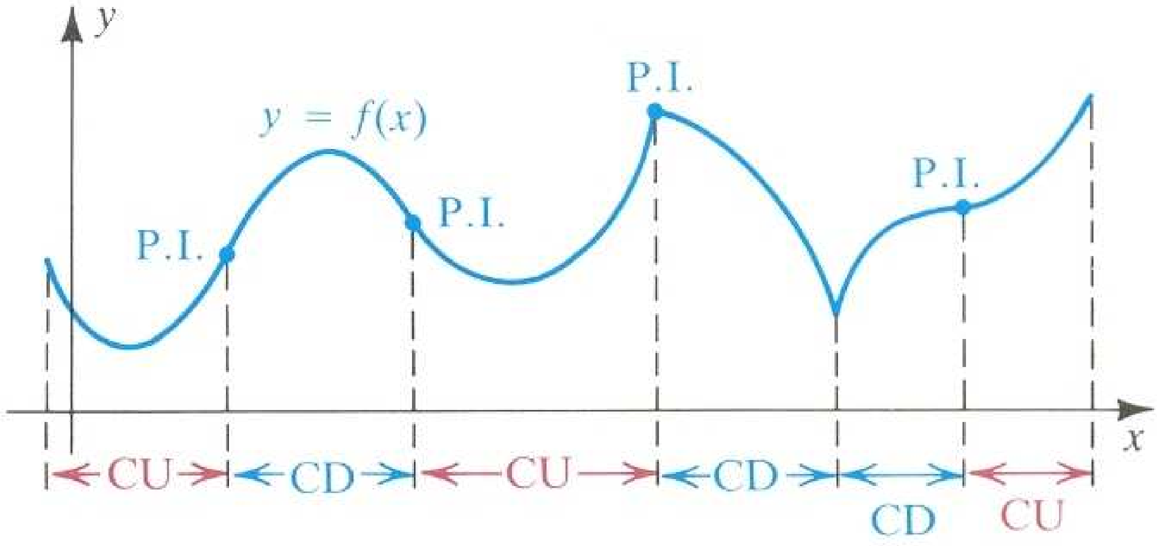
Definiyion (4.17) : ***Page (193)***

A point $\left(c,f\left(c\right)\right)$ *on the graph of* f *is a point of inflection if the following two condition are satisfied :*

(i) f *is continuous at* c *.*

(ii) There is an open interval $\left(a,b\right)$ *containing* c *such that the graph is concave upward on* $\left(a,c\right)$ *and concave downward on* $\left(c,b\right)$ *, or vice versa .*

Figure 4.31



Second derivative test (4.18) : *Page (194)*

Suppose that f is differentiable on an open interval containing c and that .

(i) If $f''(c) < 0$, then f has a local maximum at c.

(ii) If $f''(c) > 0$, then f has a local minimum at c.

Figure 4.32

local maximum $f(c)$

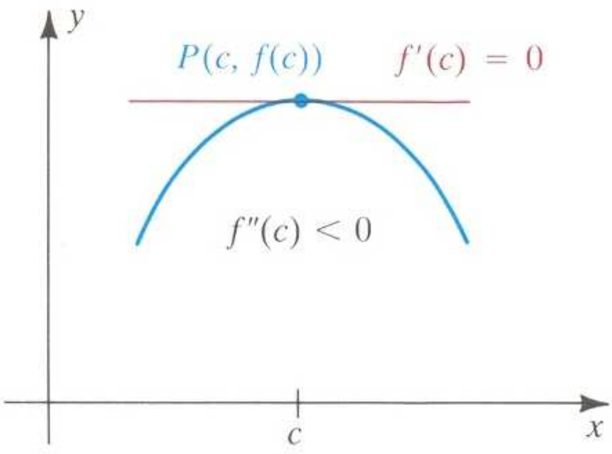
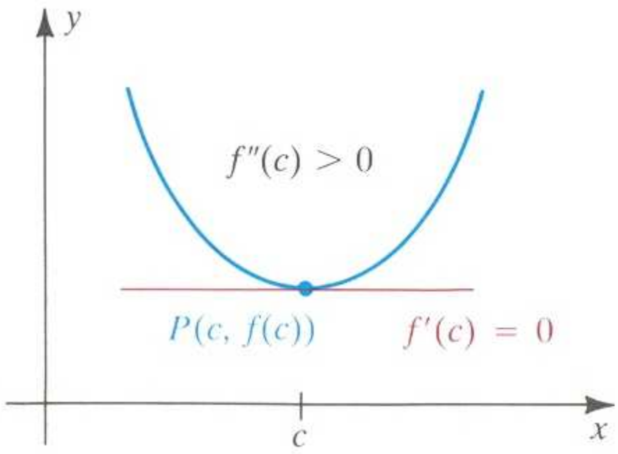


Figure 4.33

local minimum $f(c)$



Example (3) : *Page (194)*

If $f(x) = 12 + 2x^2 - x^4$, use the second derivative test to find the local extrema of f. Discuss concavity, find the points of inflection, and sketch the graph of f.

Solution

$$f(x) = 12 + 2x^2 - x^4$$

* *Differentiating* $f(x)$ *twice yields*

$$f'(x) = 4x - 4x^3 = 4x\left(1 - x^2\right)$$

$$f''(x) = 4 - 12x^2 = 4\left(1 - 3x^2\right).$$

* *The expression for* $f'(x)$ *is used to find the critical numbers* $0, 1,$ **and** -1 .

* *The values of* f'' *at these numbers are*

$$f''(0) = 4 > 0 \ , \quad f''(1) = -8 < 0 \ , \quad \textbf{and} \quad f''(0) = 4 > 0$$

* *Hence , by the second derivative test , the function has a local minimum at* 0 *and local maxima at* 1 *and* -1 .

The corresponding function values are $f(0) = 12$ *and* $f(1) = 13 = f(-1)$.

* *The following table summarizes our discussion .*

Critical number c	$f''(c)$	*Sign of* $f''(c)$	*Conclusion*
-1	-8	$-$	*local max :* $f(-1) = 13$
0	4	$+$	*local min :* $f(0) = 12$
1	-8	$-$	*local max :* $f(1) = 13$

* *To locate the possible points of inflection , we solve the equation* $f''(x) = 0$ *(that is ,* $4\left(1 - 3x^2\right) = 0$ *) , obtaining the solutions* $-3\sqrt{3}$ *and* $3\sqrt{3}$. *we next examine the sign of* $f''(x)$ *in each of the intervals*

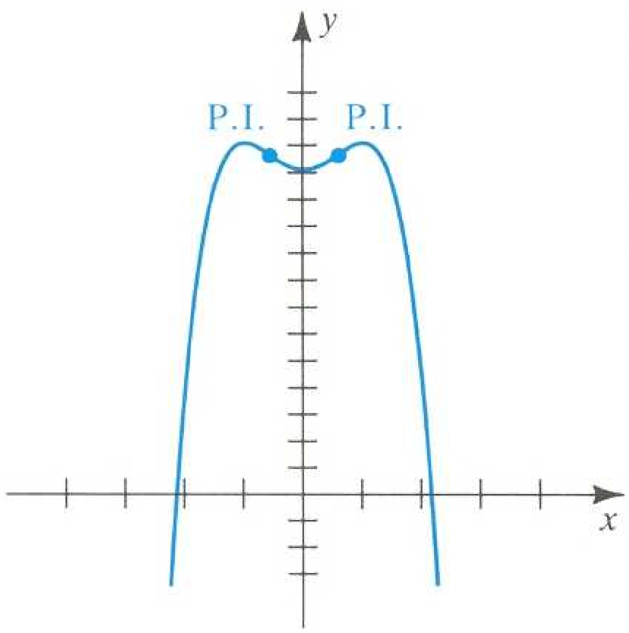
$$\left(-\infty, -\sqrt{3}/3\right), \quad \left(-\sqrt{3}/3, \sqrt{3}/3\right), \quad \textbf{and} \quad \left(\sqrt{3}/3, \infty\right).$$

* *Since* f'' *is continuous and has no zeros on each interval , we may use test values to determine the sign of* $f''(x)$. *Let us arrange our work in tabular form as follows . The last row is a consequence of (4.16) .*

Interval	$\left(-\infty, -\sqrt{3}/3\right)$	$\left(-\sqrt{3}/3, \sqrt{3}/3\right)$	$\left(\sqrt{3}/3, \infty\right)$
k	-1	0	1
Test value $f''(k)$	$f''(-1) = -8$	$f''(0) = 4$	$f''(1) = -8$
Sign of $f''(x)$	$-$	$+$	$-$
Concavity	downward	upward	downward

* *Since f'' changes sign at $-3\sqrt{3}$ and $3\sqrt{3}$, the corresponding points $\left(\pm\sqrt{3}/3, 113/9\right)$ on the graph are points of inflection. These are the points at which the concavity changes. As shown in the table, the graph is concave upward on the open interval $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$ and concave downward outside of $\left[-\sqrt{3}/3, \sqrt{3}/3\right]$. The graph is sketched in Figure 4.34.*

Figure 4.34



Example (4) : *Page (195)*

If $f(x) = x^5 - 5x^3$, find the local extrema of f. Discuss concavity, find the points of inflection, and sketch the graph of f.

Solution

$$f(x) = x^5 - 5x^3$$

* *We begin by differentiating* $f(x)$ *twice :*

$$f'(x) = 5x^4 - 15x^2 = 5x^2\left(x^2 - 3\right)$$

$$f''(x) = 20x^3 - 30x = 10x\left(2x^2 - 3\right).$$

* *Solving the equation* $f'(x) = 0$ *gives us the critical numbers* $0, -\sqrt{3}$, *and* $\sqrt{3}$. *We obtain the following table .*

Critical number c	$f''(c)$	*Sign of* $f''(c)$	*Conclusion*
$-\sqrt{3}$	$-30\sqrt{3}$	$-$	*local max :* $f\left(-\sqrt{3}\right) = 6\sqrt{3}$
0	0	*none*	*No conclusion*
$\sqrt{3}$	$30\sqrt{3}$	$+$	*local min :* $f\left(\sqrt{1}\right) = -6\sqrt{3}$

* *Since* $f''(0) = 0$, *the second derivative test is not applicable at* 0, *and so we apply the first derivative test .* *We can show , using test values , that if* $-\sqrt{3} < x < 0$, *then* $f'(x) < 0$, *and if* $0 < x < \sqrt{3}$, *then* $f'(x) < 0$. *Since* $f'(x)$ *does not change sign , there is no extremum at* $x = 0$.

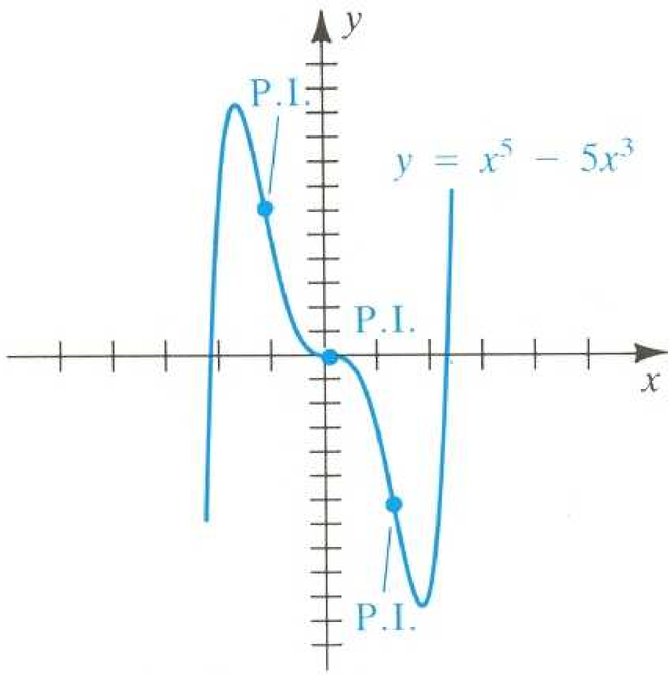
* *To find possible points of inflection , we consider the equation* $f''(x) = 0$ *- that is ,* $10x\left(2x^2 - 3\right) = 0$. *The solutions of this equation , in order of magnitude , are* $-\sqrt{6}/2, 0$, *and* $\sqrt{6}/2$. *We construct a table .*

Interval	$\left(-\infty, -\sqrt{6}/2\right)$	$\left(-\sqrt{6}/2, 0\right)$	$\left(0, \sqrt{6}/2\right)$	$\left(\sqrt{6}/2, \infty\right)$
k	-2	-1	1	2
Test value $f''(k)$	-100	10	-10	100
Sign of	$-$	$+$	$-$	$+$

$f''(x)$				
Concavity	downward	upward	downward	upward

* *The sign of* $f''(x)$ *changes at* $-\sqrt{6}/2, 0$, *and* $\sqrt{6}/2$, *so it follows that the points* $(0,0)$, $\left(-\sqrt{6}/2, 21\sqrt{6}/8\right)$, *and* $\left(\sqrt{6}/2, -21\sqrt{6}/8\right)$ *are points of inflection. The graph is sketched in Figure 4.35, with different scales on the x- and y-axes. The x-intercepts are* 0 *and* $\pm\sqrt{5} \approx 2.2$.

Figure 4.35



Example (5) : *Page (196)*

If $f(x) = 1 - x^{1/3}$, *find the local extrema. Discuss concavity, find the points of inflection, and sketch the graph of f.*

Solution

$$f(x) = 1 - x^{1/3}$$

* *Differentiating* $f(x)$ *twice yields*

$$f'(x) = -\frac{1}{3}x^{-2/3} = -\frac{1}{3x^{2/3}}$$

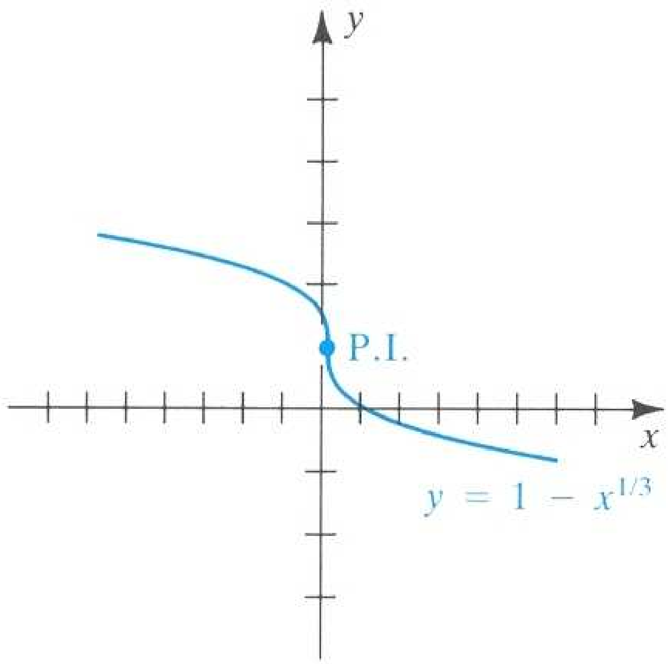
$$f''(x) = \frac{2}{9}x^{-5/3} = \frac{2}{9x^{5/3}}.$$

* *The first derivative does not exist at* $x = 0$ *, and* 0 *is the only critical number for* f *. Since* $f''(0)$ *is undefined, the second derivative test is not applicable. However, if* $x \neq 0$ *, then* $x^{2/3} > 0$ *and* $f'(x) = -1/\left(3x^{2/3}\right) < 0$ *, which means that* f *is decreasing throughout its domain. Consequently,* $f(0)$ *is not a local extremum.*
* *Using test values gives us the following table.*

Interval	$(-\infty, 0)$	$(0, \infty)$
Sign of $f''(x)$	$-$	$+$
Concavity	*downward*	*upward*

* *The concavity changes at* $x = 0$ *and* f *is continuous at* 0 *, so the point* $(0, 1)$ *is a point of inflection.*
* *The graph is sketched in Figure 4.36. Note that there is a vertical tangent line at* $(0, 1)$ *, since* f *is continuous at* $x = 0$ *and* $\lim_{x \to 0} |f'(x)| = \infty$.

Figure 4.36



Example (6) : Page (197)

If $f(x)=x^{2/3}(5+x)$, find the local extrema. Discuss concavity, find the points of inflection, and sketch the graph of f.

Solution

$$f(x)=x^{2/3}(5+x)$$

* Writing $f(x)=5x^{2/3}+x^{5/3}$ and differentiating twice gives us the following :

$$f'(x)=\frac{10}{3}x^{-1/3}+\frac{5}{3}x^{2/3}=\frac{5}{3}\left(\frac{2+x}{x^{1/3}}\right)$$

$$f''(x)=-\frac{10}{9}x^{-4/3}+\frac{10}{9}x^{-1/3}=\frac{10}{9}\left(\frac{x-1}{x^{4/3}}\right).$$

* Referring to $f'(x)$ we see that the critical numbers for f are -2 and 0. We apply the second derivative test, as indicated in the following table.

Critical number c	$(-\infty,0)$	$(0,\infty)$
Sign of $f''(c)$	$-$	none
Conclusion	Local max : $f(-2)=(-2)^{2/3}(3)\approx 4.8$	No conclusion

* Since the second derivative test is not applicable at $c=0$, let us apply the first derivative test. Using test values, we see that the sign of $f'(x)$ changes from $-$ to $+$ at $c=0$. Hence f has a local minimum at $(0,0)$.

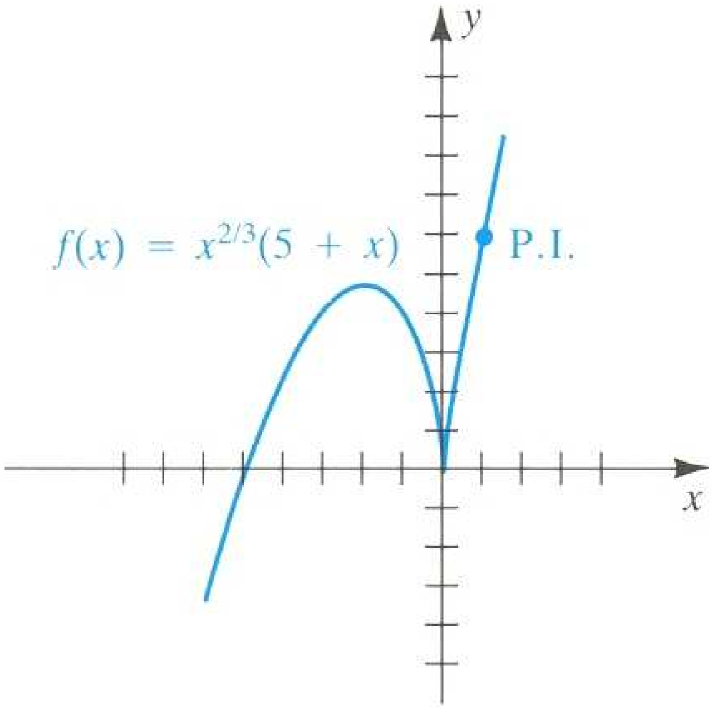
* To determine concavity, first we note that $f''(x)=0$ at $x=1$ and $f''(x)$ does not exist at $x=0$. Next we examine the sign of $f''(x)$ for the cases $x<0$, $0<x<1$, and $x>1$. Using test values for f'' leads to the following table.

Interval	$(-\infty,0)$	$(0,1)$	$(1,\infty)$
Sign of $f''(x)$	$-$	$-$	$+$

Concavity	*downward*	*downward*	*upward*

* *We see from the table that the graph of* f *has a point of inflection at* $(1,6)$ *, but not at* $(0,0)$ *. The graph is sketched in Figure 4.37 .*

Figure 4.37



* *Note that*

$$\lim_{x \to 0^-} f'(x) = -\infty \quad \text{and} \quad \lim_{x \to 0^+} f'(x) = \infty .$$

* *Since* f *is continuous at* $x = 0$ *, the graph has a cusp at* $(0,0)$ *by Definition (3.10) .*

Example (7) : ***Page (197)***

If $f(x) = 2\sin x + \cos 2x$ *, find the local extrema and sketch the graph of* f *on the interval* $[0, 2\pi]$ *.*

Solution

$$f(x) = 2\sin x + \cos 2x$$

* *We differentiate* $f(x)$ *twice :*

$$f'(x) = 2\cos x - 2\sin 2x$$

$$f''(x) = -2\sin x - 4\cos 2x \ .$$

* *In Example (7) of Section 4.1 we found that the critical numbers of f in the interval* $[0, 2\pi]$ *are* $\pi/6$, $5\pi/6$, $\pi/2$, *and* $3\pi/2$.

* *Substituting these critical numbers for x in* $f''(x)$ *, we obtain*

$$f''(\pi/6) = -3 \quad , \quad f''(5\pi/6) = -3 \ ,$$

$$f''(\pi/2) = 2 \quad , \quad f''(3\pi/2) = 6 \ .$$

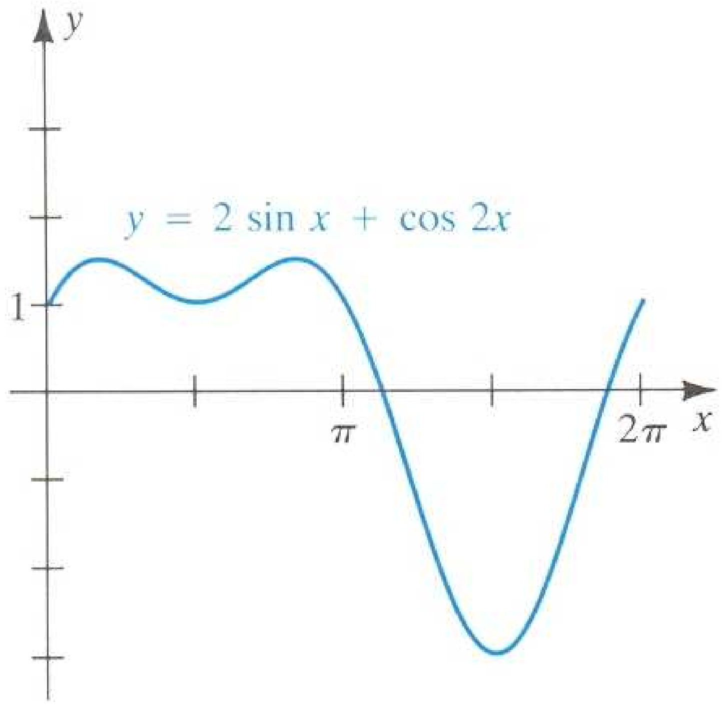
* *Applying the second derivative test , we see that there are local maxima at* $\pi/6$ *and* $5\pi/6$ *and local minima at* $\pi/2$ *and* $3\pi/2$. *Thus , we have*

$$\text{Local max}: f(\pi/6) = 3/2 \quad \text{and} \quad f(5\pi/6) = 3/2$$

$$\text{Local min}: f(\pi/2) = 1 \qquad \text{and} \quad f(3\pi/2) = -3$$

* *Using this information and plotting several more points gives us the sketch in Figure 4.38 .*

Figure 4.38



Exercises 4.4: 13, 19, 25, 31.

<u>4.5 SUMMARY OF GRAPHICAL METHODS :</u> *Page (199)*

<u>Guidelines for sketching the graph of</u> $y = f(x)$ ***<u>(4.19) :</u>***

Page (200)

1 Domain of f

Find the domain of f - that is, all real numbers x such that $f(x)$ is defined.

2 Concavity of f

Determine whether f is continuous on its domain, and, if not, find and classify the discontinuities.

3 x- and y-intercepts

The x-intercepts are the solutions of the equation $f(x)=0$; the y-intercept is the function value $f(0)$, if it exists.

4 Symmetry

If f is an even function, the graph is symmetric with respect to the y-axis. If f is an odd function, the graph is symmetric with respect to the origin.

5 Critical numbers and local extrema

Find $f'(x)$ and determine the critical numbers – that is, the values of x such that $f'(x)=0$ or $f'(x)$ does not exist. Use the first derivative test to help find local extrema. Employ the sign of $f'(x)$ to find intervals on which f is increasing $\left(f'(x)>0\right)$ or is decreasing $\left(f'(x)<0\right)$. Determine whether there are corners or cusps on the graph.

6 Concavity and points of inflection

Find $f''(x)$, and use the second derivative test whenever appropriate. If $f''(x)>0$ on an open interval I, the graph is concave upward. If $f''(x)<0$, the graph is concave downward. If f is continuous at c and if $f''(x)$ changes sign at c, then $P\left(c, f(c)\right)$ is a point of inflection.

7 Asymptotes

Horizontal : If $\lim\limits_{x\to\infty} f(x)=L$ or $\lim\limits_{x\to-\infty} f(x)=L$, then the line $y=L$ is a horizontal asymptote.

Vertical : If $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to a^-} f(x)$ is either ∞ or $-\infty$, then the line $x=a$ is a horizontal asymptote.

Example (1) : *Page (200)*

If $f(x)=\dfrac{2x^2}{9-x^2}$ *, discuss and sketch the graph of* f *.*

Solution

$$f(x)=\frac{2x^2}{9-x^2}$$

We shall follow Guidelines (4.19) .

** Guideline 1*

The domain of f *consists of all real numbers except* -3 *and* 3 *.*

** Guideline 2*

The function f *has infinite discontinuities at* -3 *and* 3 *and is continuous at all other real numbers .*

** Guideline 3*

To find the x-intercepts , we solve the equation $f(x)=0$ *, obtaining* $x=0$ *. The y-intercept is* $f(0)=0$ *. Therefore , the graph intersects both the x-axis and the y-axis at the origin .*

** Guideline 4*

Since $f(-x)=f(x)$ *,* f *is an even function and the graph is symmetric with respect to the y-axis .*

** Guideline 5*

We differentiate $f(x)$ *:*

Remember that :

* $$D_x\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)D_xf(x)-f(x)D_xg(x)}{\left[g(x)\right]^2}$$

$$f'(x)=\frac{(9-x^2)(4x)-2x^2(-2x)}{(9-x^2)^2}=\frac{36x}{(9-x^2)^2}$$

Since $f'(x) = 0$ if $x = 0$, 0 is a critical number . The numbers -3 and 3 are not critical numbers because they are not in the domain of f .

Using test values gives us the following table .

Interval	$(-\infty, -3)$	$(-3, 0)$	$(0, 3)$	$(3, \infty)$
Sign of $f'(x)$	$-$	$-$	$+$	$+$
Conclusion	f is decreasing	f is decreasing	f is increasing	f is increasing

The sign of the derivative $f'(x)$ changes from negative to positive at $x = 0$, so , by the first derivative test , $f(0) = 0$ is a local minimum for f .

* Guideline 6

We differentiate $f'(x)$:

$$f''(x) = \frac{(9 - x^2)^2 (36) - (36x)(2)(9 - x^2)(-2x)}{(9 - x^2)^4}$$

$$= \frac{108(x^2 + 3)}{(9 - x^2)^3}$$

The numerator is always positive , so the sign of $f''(x)$ is determined by $(9 - x^2)^3$. Using test values gives us the following .

Interval	$(-\infty, -3)$	$(-3, 3)$	$(3, \infty)$
Sign of $f''(x)$	$-$	$+$	$-$
Concavity	downward	upward	downward

Since f is not continuous at -3 or 3, there are no points of inflection. As a check on local extrema (see guideline 5), we note that $f''(x)>0$, and hence, by the second derivative test, $f(0)=0$ is a local minimum.

*** Guideline 7**

To find horizontal asymptotes, we use the methods in Section 2.4, obtaining

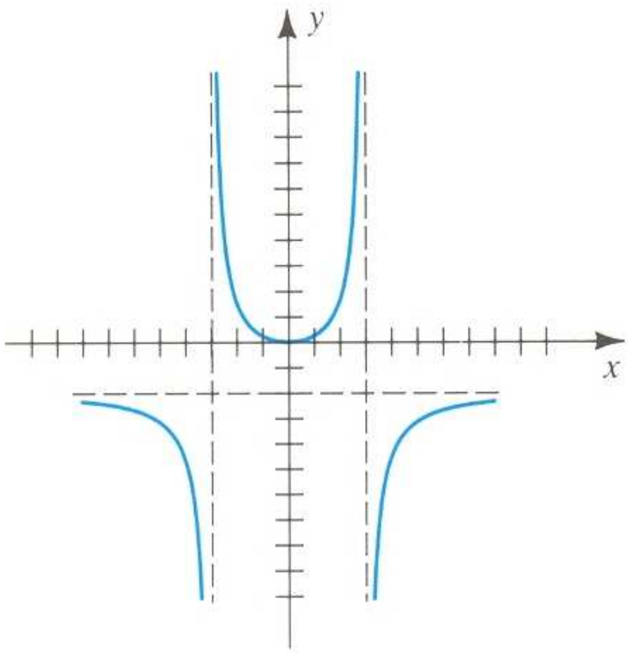
$$\lim_{x\to\infty}\frac{2x^2}{9-x^2}=-2 \quad \text{and} \quad \lim_{x\to-\infty}\frac{2x^2}{9-x^2}=-2$$

Thus, the line $y=-2$ is a horizontal asymptote.

The vertical asymptotes correspond to the zeros of the denominator $9-x^2$ and hence are $x=-3$ and $x=3$.

Using the results of the guidelines and referring to the table developed from guideline 5 to obtain the behavior of f near the vertical asymptotes $(x=\pm 3)$ gives us the sketch in Figure 4.40.

Figure 4.40



Example (2) : *Page (202)*

If $f(x)=\dfrac{x^2}{x^2-x-2}$, discuss and sketch the graph of f.

Solution

$$f(x)=\frac{x^2}{x^2-x-2}$$

We shall follow Guidelines (4.19) .

* Guideline 1

The denominator equals $(x-2)(x+1)$, so the domain of f consists of all numbers except -1 and 2 .

* Guideline 2

The function f has infinite discontinuities at -1 and 2 and is continuous at all other real numbers .

* Guideline 3

To find the x-intercepts , we solve the equation $f(x)=0$, obtaining $x=0$. The y-intercept is $f(0)=0$. Therefore , the graph intersects both the x-axis and the y-axis at the origin .

* Guideline 4

Since f is neither even nor odd , the graph is not symmetric to the y-axis or to the origin .

* Guideline 5

We differentiate $f(x)$:

Remember that :

$$* \quad D_x\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)D_xf(x)-f(x)D_xg(x)}{[g(x)]^2}$$

$$f'(x)=\frac{(x^2-x-2)(2x)-x^2(2x-1)}{(x^2-x-2)^2}=-\frac{x(x+4)}{(x^2-x-2)^2}$$

Since $f'(x)=0$ gives us the critical numbers 0 and -4 . The zeros of the denominator , -1 and 2 are not critical numbers since $f(-1)$ and $f(2)$ do not exist .

Using test values gives us the following table .

Interval	$(-\infty,-4)$	$(-4,-1)$	$(-1,0)$	$(0,2)$	$(2,\infty)$

Sign of $f'(x)$	$-$	$+$	$+$	$-$	$-$
Conclusion	f is decreasing	f is increasing	f is increasing	f is decreasing	f is decreasing

By the first derivative test, f has a local minimum $f(-4)=\frac{8}{9}$ and a local maximum $f(0)=0$.

*** Guideline 6**

We will not discuss concavity or find points of inflection. You may wish to verify that

$$f''(x)=\frac{2\left(x^3+6x^2+4\right)}{\left(x^2-x-2\right)^3}.$$

To solve the equation $f''(x)=0$, we must solve the cubic equation

$$x^3+6x^2+4=0.$$

It can be shown that this equation has one real root r. To the nearest tenth, $r\approx-6.1$. The point of inflection is approximately $(-6.1, 0.9)$ slightly higher than the low (minimum) point $\left(-4,\frac{8}{9}\right)$.

*** Guideline 7**

To find horizontal asymptotes, we consider

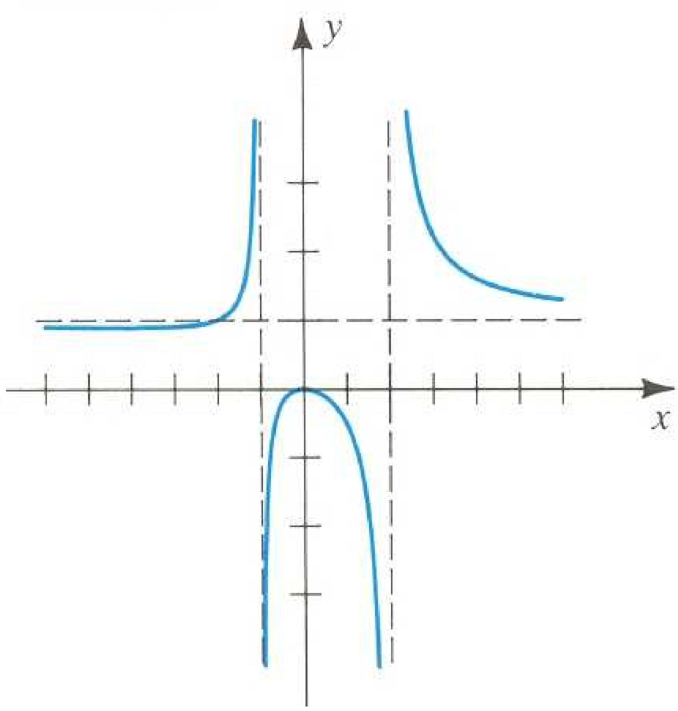
$$\lim_{x\to\infty}\frac{x^2}{x^2-x-2}=1 \quad \text{and} \quad \lim_{x\to-\infty}\frac{x^2}{x^2-x-2}=1$$

Thus, the graph has a horizontal asymptote $y=1$.

The vertical asymptotes are obtained from the solusion of the equation $x^2-x-2=0$, so they are $x=-1$ and $x=2$.

Using the preceding results and referring to the table developed from guideline 5 to obtain the behavior of f near the vertical asymptotes leads to the sketch in Figure 4.41.

Figure 4.41



The graph intersects the horizontal asymptote $y = 1$. *To find the x-coordinate of the point of inflection, we solve the equation* $f(x) = 1$ *as follows :*

$$\frac{x^2}{x^2 - x - 2} = 1$$

$$x^2 = x^2 - x - 2$$

$$x = -2$$

Hence the point of inflection is $(-2, 1)$.

Example (3) : *Page (203)*

If $f(x) = \dfrac{x^2 - 9}{2x - 4}$ *, discuss and sketch the graph of* f.

Solution

$$f(x) = \frac{x^2 - 9}{2x - 4}$$

We shall follow Guidelines (4.19) .

*** Guideline 1 and 2**

The domain of f *consists of all real numbers except* $x = 2$ *, where there is an infinite discontinuity .*

*** Guideline 3**

The x-intercepts are -3 and 3, and the y-intercept is $f(0)=\frac{9}{4}$.

*** Guideline 4**

The graph is symmetric with respect to neither the y-axis nor the origin.

*** Guideline 5 and 6**

You should verify that

Remember that :

$$* \quad D_x\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)D_xf(x)-f(x)D_xg(x)}{[g(x)]^2}$$

$$f'(x)=\frac{x^2-4x+9}{2(x-2)^2} \quad \text{and} \quad f''(x)=-\frac{5}{(x-2)^3}$$

Since $f'(x)\neq 0$ for every $x\neq 2$, there is no local extrema (see Corollary (4.6)).

Since $f''(x)>0$ if $x<2$ and $f''(x)<0$ if $x>2$, the graph of f is concave upward on $(-\infty,2)$ and concave downward on $(2,\infty)$. There is no point of inflection

*** Guideline 7**

The degree of the numerator x^2-9 is greater than that of the denominator $2x-4$, so there is an oblique asymptote and we use long division to express $f(x)$ as follows :

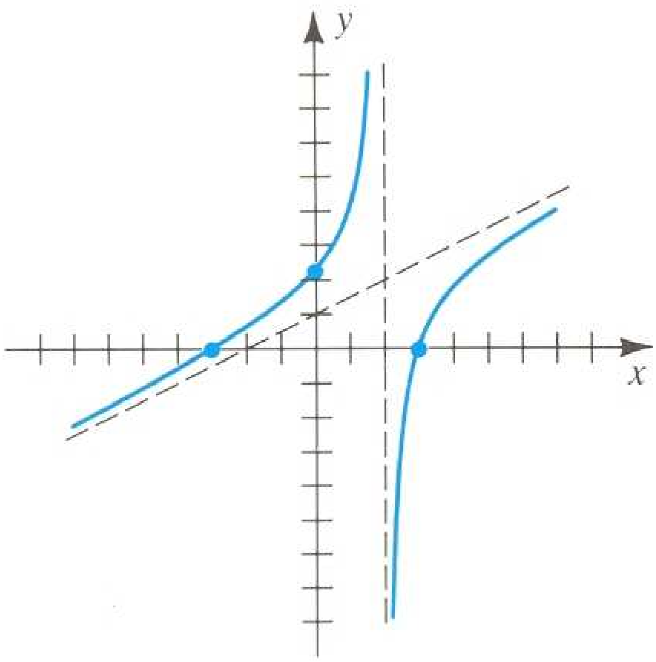
$$\frac{x^2-9}{2x-4}=\left(\frac{1}{2}x+1\right)-\frac{5}{2x-4}$$

From the discussion preceding this example, the line $y=\frac{1}{2}x+1$ is an oblique asymptote.

Since the graph has an oblique asymptote, there is no horizontal asymptote. There is a vertical asymptote $x=2$ corresponding to the zero of the denominator $2x-4$.

Representing the asymptotes by dashed lines, plotting the intercepts, and using the other information obtained by following the guidelines gives us the sketch in Figure 4.42.

Figure 4.42



Exercises 4.5: 15, 19, 3

CHAPTER (5)

INTEGRALS

5.1 ANTIDERIVATIVE AND INDEFINITE INTEGRALS :

Page (240)

Definition (5.1) : *Page (240)*

A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for every x in I.

* *We shall also call $F(x)$ an antiderivative of $f(x)$. The process of finding F, or $F(x)$, is called antidifferentiation.*

* *To illustrate, $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, because*

$$F'(x) = D_x\left(x^2\right) = 2x = f(x).$$

* *There are many other antiderivatives of $2x$, such as x^2+2, $x^2+\frac{5}{3}$, $x^2+\sqrt{3}$. In general, If C is any constant, then is an antiderivative of $2x$, because*

$$D_x\left(x^2+C\right) = 2x+0 = 2x.$$

* *Thus, there is a family of antiderivatives of $2x$ of the form $F(x) = x^2+C$, where C is any constant. Graphs of several members of this family are sketched in Figure 5.1.*

Figure 5.1

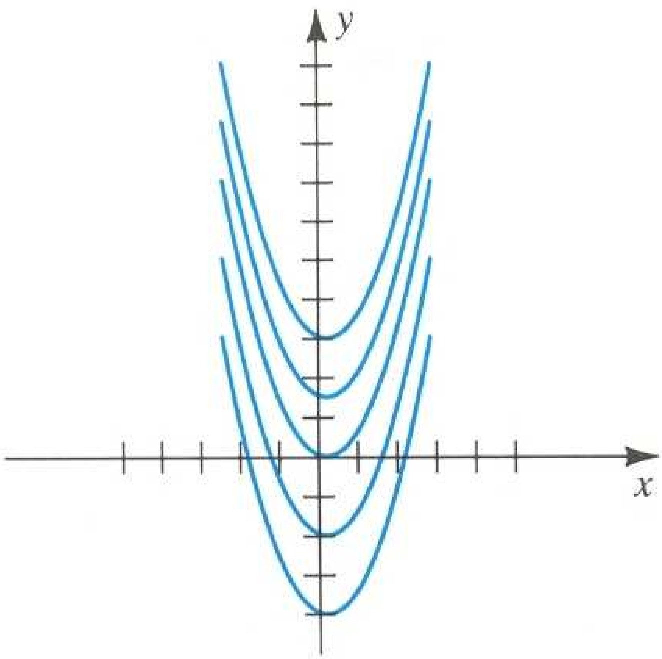


ILLUSTRATION : *Page (240)*

$f(x)$	*Antiderivative of* $f(x)$
* x^2	$\frac{1}{3}x^3$, $\frac{1}{3}x^3+8$, $\frac{1}{3}x^3+C$
* $8x^3$	$2x^4$, $2x^4-\sqrt[3]{7}$, $2x^4+C$
* $\cos x$	$\sin x$, $\sin x+\frac{4}{9}$, $\sin x+C$

Definition (5.2) : *Page (240)*

Let F be an antiderivative of f on an interval I. If G is any antiderivative of f on I, then

$$G(x)=F(x)+C$$

For some constant C and every x in I.

Definition (5.3) : *Page (241)*

The notation

$$\int f(x)\,dx = F(x)+C$$

where $F'(x)=f(x)$ *and* C *is an arbitrary constant , denotes the family of all antiderivatives of on an interval* I .

Brief table of indefinite integrals (5.4) : ***Page (242)***

Derivative $D_x\left[f(x)\right]$	*Indefinite integral* $\int D_x\left[f(x)\right]dx = f(x)+C$
$D_x(x)=1$	$(1)\ \int 1\,dx = \int dx = x+C$
$D_x\left(\dfrac{x^{r+1}}{r+1}\right)=x^{r+1}\quad (r\neq -1)$	$(2)\ \int x^r\,dx = \dfrac{x^{r+1}}{r+1}+C\quad (r\neq -1)$
$D_x(\sin x)=\cos x$	$(3)\ \int \cos x\,dx = \sin x + C$
$D_x(-\cos x)=\sin x$	$(4)\ \int \sin x\,dx = -\cos x + C$
$D_x(\tan x)=\sec^2 x$	$(5)\ \int \sec^2 x\,dx = \tan x + C$
$D_x(-\cot x)=\csc^2 x$	$(6)\ \int \csc^2 x\,dx = -\cot x + C$
$D_x(\sec x)=\sec x\tan x$	$(7)\ \int \sec x\tan x\,dx = \sec x + C$
$D_x(-\csc x)=\csc x\cot x$	$(8)\ \int \csc x\cot x\,dx = -\csc x + C$

ILLUSTRATION : ***Page (242)***

* $\int x^3 . x^5\,dx = \int x^8\,dx = \dfrac{x^{8+1}}{8+1} = \boxed{\dfrac{1}{9}x^9+C}$.
* $\int \dfrac{1}{x^3}\,dx = \int x^{-3}\,dx = \dfrac{x^{-3+1}}{-3+1} = \boxed{-\dfrac{1}{2x^2}+C}$.

$$* \int \sqrt[3]{x^2}\, dx = \int x^{2/3}\, dx = \frac{x^{2/3+1}}{\frac{2}{3}+1} = \boxed{\frac{3}{5} x^{5/3} + C}.$$

$$* \int \frac{\tan x}{\sec x}\, dx = \int \cos x \, \frac{\sin x}{\cos x} dx = \int \sin x \, dx = \boxed{-\cos x + C}.$$

Theorem (5.5) : *Page (243)*

$$\text{(i)} \int \left[D_x f(x) \right] dx = f(x) + C .$$

$$\text{(ii)} \ D_x \left[\int f(x)\, dx \right] = f(x) .$$

Example (1) : *Page (243)*

Verify Theorem (5.5) for the special case $f(x) = x^2$.

Solution

$$f(x) = x^2$$

(i) If we first differentiate x^2 *and then integrate ,*

$$\int \left[D_x \left(x^2 \right) \right] dx = \int 2x \, dx = \boxed{x^2 + C}.$$

(ii) If we first integrate x^2 *and then differentiate ,*

$$D_x \int x^2 \, dx = D_x \left(\frac{x^3}{3} + C \right) = \boxed{x^2}.$$

Theorem (5.6) : *Page (243)*

$$\text{(i)} \int c\, f(x)\, dx = c \int f(x)\, dx \quad \textit{for every constant } c .$$

$$\text{(ii)} \int \left[f(x) + g(x) \right] dx = \int f(x)\, dx + \int g(x)\, dx .$$

$$\text{(iii)} \int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx \,.$$

Example (2) : *Page (244)*

Evaluate $\int \left(5x^3 + 2\cos x\right) dx$.

Solution

* *We first use (ii) and (i) of Theorem (5.6) and then formulas from (5.4) :*

$$\int \left(5x^3 + 2\cos x\right) dx = \int 5x^3\,dx + \int 2\cos x\,dx$$

$$= 5\int x^3\,dx + 2\int \cos x\,dx$$

$$= 5\left(\frac{x^4}{4} + C_1\right) + 2\left(\sin x + C_2\right)$$

$$= \frac{5}{4}x^4 + 5C_1 + 2\sin x + 2C_2$$

$$= \boxed{\frac{5}{4}x^4 + 2\sin x + C} \,.$$

where $C = 5C_1 + 2C_2$.

Example (3) : *Page (244)*

Evaluate $\int \left(8t^3 - 6\sqrt{t} + \frac{1}{t^3}\right) dt$.

Solution

* *First we find an antiderivative for each of the three terms in the integral and then add an arbitrary constant* C. *We rewrite* $\sqrt{t}$ *as* $t^{1/2}$ *and* $1/t^3$ *as* t^{-3} *and the use the power rule for integration :*

Remember that :

* $\boxed{D_x x^n = nx^{n-1}}$

$$\int\left(8t^{3}-6\sqrt{t}+\frac{1}{t^{3}}\right)dt=\int\left(8t^{3}-6t^{1/2}\sqrt{t}+t^{-3}\right)dt$$

$$=8\frac{t^{4}}{4}-6\left(\frac{t^{3/2}}{3/2}\right)+\frac{t^{-2}}{-2}+C$$

$$=\boxed{2t^{4}-4t^{3/2}-\frac{1}{2t^{2}}+C}.$$

Example (4) : *Page (244)*

Evaluate $\displaystyle\int\frac{\left(x^{2}-1\right)^{2}}{x^{2}}dx.$

Solution

* *First we change the form of the integrand , because the degree of the numerator is greater than or equal to the degree of the denominator . We then find an antiderivative for each term , adding an arbitrary constant C after last integration :*

$$\int\frac{\left(x^{2}-1\right)^{2}}{x^{2}}dx=\int\frac{x^{4}-2x^{2}+1}{x^{2}}dx$$

$$=\int\left(x^{2}-2+x^{-2}\right)dx$$

$$=\frac{x^{3}}{3}-2x+\frac{x^{-1}}{-1}+C$$

$$=\boxed{\frac{1}{3}x^{3}-2x-\frac{1}{x}+C}.$$

Example (5) : *Page (245)*

Evaluate $\displaystyle\int\frac{1}{\cos u\,\cot u}du.$

Solution

* *We use trigonometric identities to change the integrad and the apply formula (7) from Table (5.4) :*

$$\int \frac{1}{\cos u \cot u}\, du = \int \sec u \tan u \, du$$

$$= \boxed{\sec u + C}.$$

Exercises 5.1: 3, 7, 19, 21, 27, 41.

5.2 CHANGE OF VARIABLES IN INDEFINITE INTEGRALS:

Page (250)

Definition (5.7) : *Page (251)*

If F is an antiderivative of f ,

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C .$$

If $\boxed{u = g(x)}$ *and* $du = g'(x)\, dx$ *, then*

$$\int f(u)\, du = F(u) + C .$$

Remember that :

$$* \quad \frac{d}{dx} F(g(x)) = f(g(x))\, g'(x)$$

Example (1) : *Page (252)*

Evaluate $\int \sqrt{5x+7}\, dx$.

Solution

$$\int \sqrt{5x+7}\, dx$$

* *The integral may be written as in Formulas from (5.4) (2) by using the substitution*

$$\boxed{u = 5x + 7}$$

$$du = 5\, dx \quad \Rightarrow \quad \frac{1}{5} du = dx$$

* *We proceed as follows :*

$$\int \sqrt{5x+7}\, dx = \frac{1}{5}\int \sqrt{u}\, du$$

$$= \frac{1}{5}\int u^{1/2}\, du$$

Remember that :

$$* \quad \int x^r\, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{1}{5}\left(\frac{u^{3/2}}{3/2}\right) + C$$

$$= \frac{2}{15} u^{3/2} + C$$

$$= \boxed{\frac{2}{15}(5x+7)^{3/2} + C}.$$

Example (2) : Page (252)

Evaluate $\int \cos 4x\, dx$.

Solution

$$\int \cos 4x\, dx$$

* The integral may be written as in Formulas from (5.4) (3) by using the substitution

$$\boxed{u = 4x}$$

$$du = 4\, dx \quad \Rightarrow \quad \frac{1}{4} du = dx$$

* We proceed as follows :

$$\int \cos 4x\, dx = \frac{1}{4}\int \cos u\, du$$

Remember that :

$$* \quad \boxed{\int \cos dx = \sin x + C}$$

$$= \frac{1}{4} \sin u + C$$

$$= \boxed{\frac{1}{4} \sin 4x + C}.$$

<u>**Guidelines for changing variables in indefinite integrals (5.8) :**</u>

Page (253)

1 Decide on a reasonable substation $u = g(x)$.

2 Calculate $du = g'(x)\,dx$.

3 Using 1 and 2 , try to transform the integral into a form that involves only the variable u . If necessary , introduce a constant factor k into the integrand and compensate by multiplying the integral by $1/k$. If any part of the resulting integrand contains the variable x , use a different substitution in 1 .

4 Evaluate the integral obtained in 3 , obtaining an antiderivative involving u .

5 Replace u in the antiderivative obtained in guideline 4 by $g(x)$. The final result should contain only the variable x .

<u>**Example (3) :**</u> **Page (253)**

Evaluate $\int \left(2x^3 + 1\right)^7 x^2 \, dx$.

Solution

$$\int \left(2x^3 + 1\right)^7 x^2 \, dx$$

* The integral may be written as in Formulas from (5.4) (2) by using the substitution

$$\boxed{u = 2x^2 + 1}$$

$$du = 6x^2 dx \quad \Rightarrow \quad \frac{1}{6} du = x^2 dx$$

* We proceed as follows :

$$\int \left(2x^3+1\right)^7 x^2\, dx = \frac{1}{6}\int u^7\, du$$

<u>Remember that :</u>

$$* \quad \int x^r\, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{1}{6}\left(\frac{u^8}{8}\right) + C$$

$$= \boxed{\frac{1}{48}\left(2x^3+1\right)^8 + C}\,.$$

<u>Example (4) :</u> Page (254)

Evaluate $\int x\sqrt[3]{7-6x^2}\, dx$.

Solution

$$\int x\sqrt[3]{7-6x^2}\, dx$$

* The integral may be written as in Formulas from (5.4) (2) by using the substitution

$$\boxed{u = 7 - 6x^2}$$

$$du = -12\, x\, dx \quad \Rightarrow \quad -\frac{1}{12} du = x\, dx$$

* We proceed as follows :

$$\int x\sqrt[3]{7-6x^2}\, dx = -\frac{1}{12}\int \sqrt[3]{u}\, du$$

$$= -\frac{1}{12}\int u^{1/3}\, du$$

<u>Remember that :</u>

$$* \quad \int x^r\, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= -\frac{1}{12}\left(\frac{u^{4/3}}{4/3}\right) + C$$

$$= -\frac{1}{16} u^{4/3} + C$$

$$= \boxed{-\frac{1}{16}\left(7 - 6x^2\right)^{4/3} + C}.$$

Example (5) : *Page (254)*

Evaluate $\int \frac{x^2 - 1}{\left(x^3 - 3x + 1\right)^6}\, dx.$

Solution

$$\int \frac{x^2 - 1}{\left(x^3 - 3x + 1\right)^6}\, dx$$

* *The integral may be written as in Formulas from (5.4) (2) by using the substitution*

$$\boxed{u = x^3 - 3x + 1}$$

$$du = \left(3x^2 - 3\right) dx$$

$$du = 3\left(x^2 - 1\right) dx \quad \Rightarrow \quad \frac{1}{3} du = \left(x^2 - 1\right) dx$$

* *We proceed as follows :*

$$\int \frac{x^2 - 1}{\left(x^3 - 3x + 1\right)^6}\, dx = \frac{1}{3} \int \frac{1}{u^6}\, du$$

$$= \frac{1}{3} \int u^{-6}\, du$$

Remember that :

$$* \boxed{\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)}$$

$$= \frac{1}{3}\left(\frac{u^{-5}}{-5}\right) + C$$

$$= -\frac{1}{15}\left(\frac{1}{u^5}\right) + C$$

$$= \boxed{-\frac{1}{15}\frac{1}{\left(x^3 - 3x + 1\right)^5} + C}.$$

Example (6) : *Page (254)*

Evaluate $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$.

Solution

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$$

* *The integral may be written as in Formulas from (5.4) (3) by using the substitution*

$$\boxed{u = \sqrt{x}}$$

$$du = \frac{1}{2\sqrt{x}}dx \quad \Rightarrow \quad 2du = \frac{1}{\sqrt{x}}dx$$

* *We proceed as follows :*

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos u\,du$$

Remember that :

$$* \quad \boxed{\int \cos dx = \sin x + C}$$

$$= 2 \sin u + C$$

$$= \boxed{2 \sin \sqrt{x} + C}.$$

Example (7) : *Page (255)*

Evaluate $\int \cos^3 5x \ \sin 5x \ dx$.

Solution

$$\int \cos^3 5x \ \sin 5x \ dx = \int (\cos 5x)^3 \ \sin 5x \ dx$$

** The integral may be written as in Formulas from (5.4) (2) by using the substitution*

$$\boxed{u = \cos 5x}$$

$$du = -5 \sin 5x \, dx \quad \Rightarrow \quad -\frac{1}{5} du = \sin 5x \, dx$$

** We proceed as follows :*

$$\int \cos^3 5x \ \sin 5x \ dx = -\frac{1}{5} \int u^3 \ du$$

Remember that :

$$* \quad \boxed{\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)}$$

$$= -\frac{1}{5}\left(\frac{u^4}{4}\right) + C$$

$$= \boxed{-\frac{1}{20} \cos^4 5x + C}.$$

Exercises 5.2: **1, 5, 7, 13, 25.**

5.4 THE DEFINITE INTEGRAL : *Page (266)*

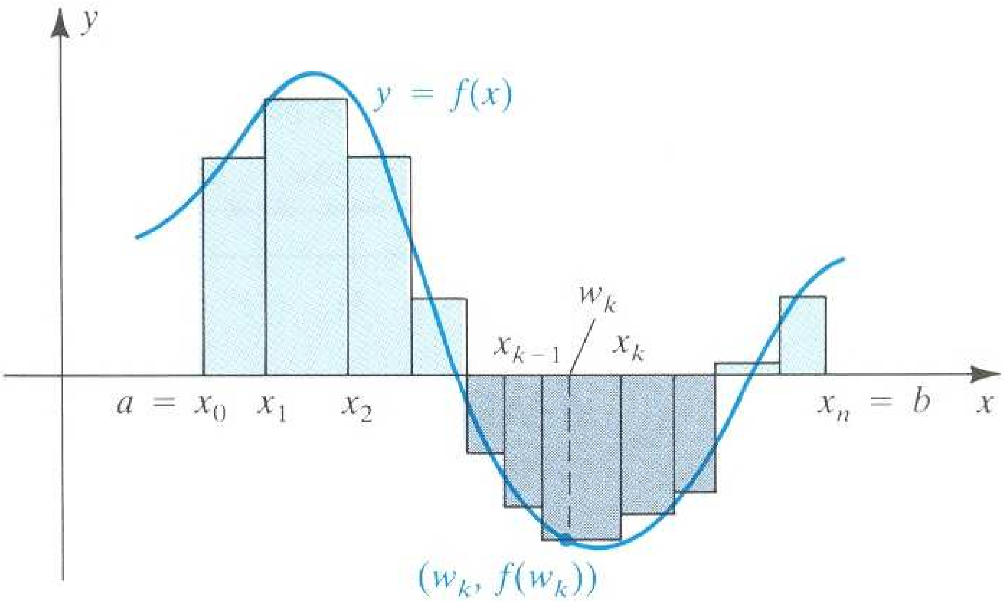
Definition (5.16) : *Page (270)*

Let f be defined on a closed interval $[a,b]$. The defined integral of f from a to b, denoted by $\int_a^b f(x)\,dx$, is

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_k f(w_k)\,\Delta x_k ,$$

provided the limit exists .

Figure 5.14



Definition (5.17) : *Page (271)*

If $c > d$, then $\int_c^d f(x)\,dx = -\int_d^c f(x)\,dx$.

Definition (5.18) : *Page (271)*

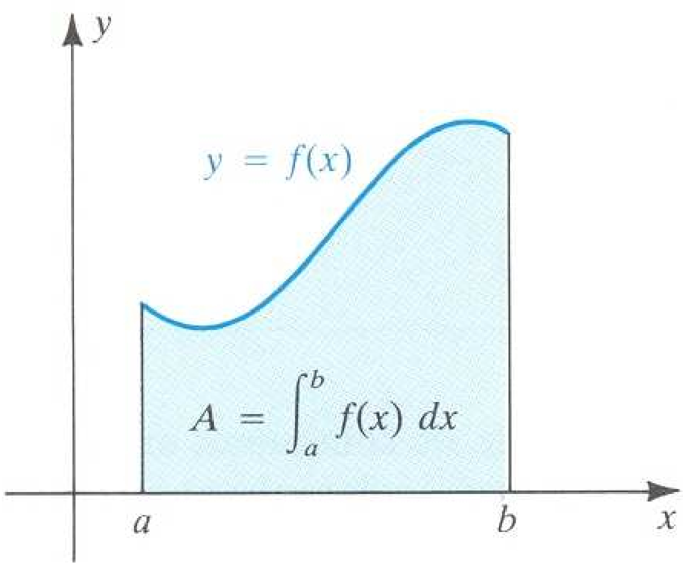
If $f(a)$, then $\int_a^a f(x)\,dx = 0$.

Definition (5.19) : *Page (271)*

If f is integrable and $f(x) \geq 0$ for every x in $[a,b]$, Then the area A of the region under the graph of f from a to b is

$$A = \int_a^b f(x)\,dx \ .$$

Figure 5.16



Exercises 5.4: 1, 5 and 7.

5.5 PROPERTIES OF THE DEFENITE INTEGRAL :

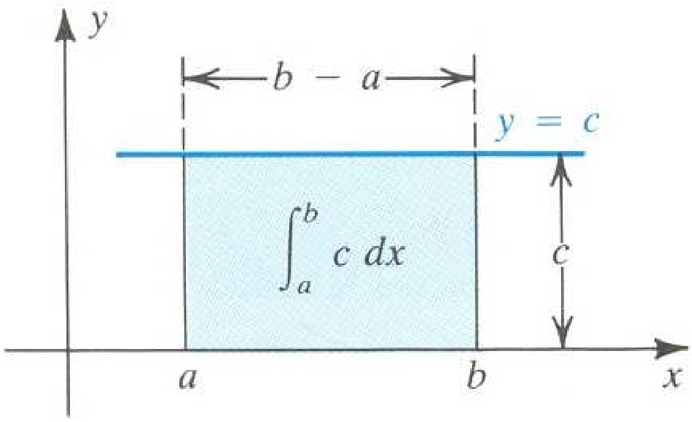
Page (275)

Theorem (5.21) : ***Page (275)***

If c is a real number , then

$$\int_a^b c\,dx = c(b-a) \ .$$

Figure 5.21



Example (1) : Page (275)

Evaluate $\int_{-2}^{3} 7\,dx$.

Solution

* Using Theorem (5.21) yields

$$\int_{-2}^{3} 7\,dx = 7\left[3-(-2)\right] = 7\,(5) = \boxed{35}\,.$$

Theorem (5.22) : Page (276)

If f is integrable on $[a,b]$ and c is any real number , then cf is integrable on $[a,b]$ and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx\,.$$

Theorem (5.23) : Page (276)

If f and g are integrable on $[a,b]$, then $f+g$ and $f-g$ are integrable on $[a,b]$ and

(i) $$\int_a^b \left[f(x)+g(x)\right]dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\,.$$

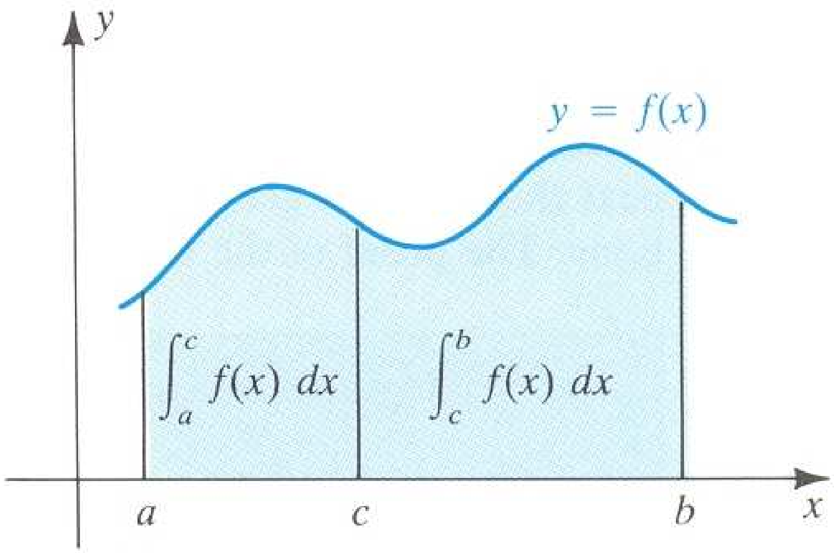
(ii) $\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$.

Theorem (5.24) : *Page (277)*

If $a < c < b$ and if f is integrable on $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$ and

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx .$$

Figure 5.22



Theorem (5.25) : *Page (277)*

If f is integrable on a closed interval and if a, b, and c are any three numbers in the interval, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx .$$

Theorem (5.26) : *Page (278)*

If f is integrable on $[a,b]$ and $f(x) \ge 0$ for every x in $[a,b]$, then

$$\int_a^b f(x)\,dx \geq 0 .$$

<u>Theorem (5.27) :</u> Page (278)

If f and g are integrable on $[a,b]$ and $f(x) \geq g(x)$ for every x in $[a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx .$$

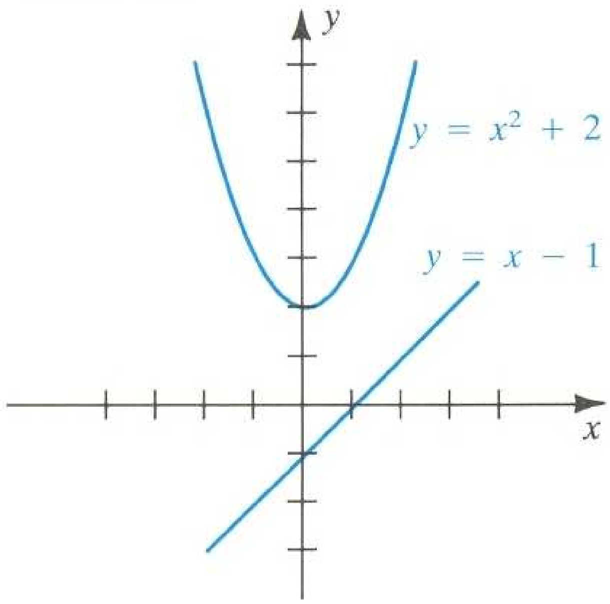
<u>Example (4) :</u> Page (278)

Show that $\displaystyle\int_{-1}^{2} \left(x^2+2\right) dx \geq \int_{-1}^{2} (x-1)\,dx$.

Solution

* The graphs of $y = x^2 + 2$ and $y = x - 1$ are sketched in Figure 5.23 .

Figure 5.23



* Since

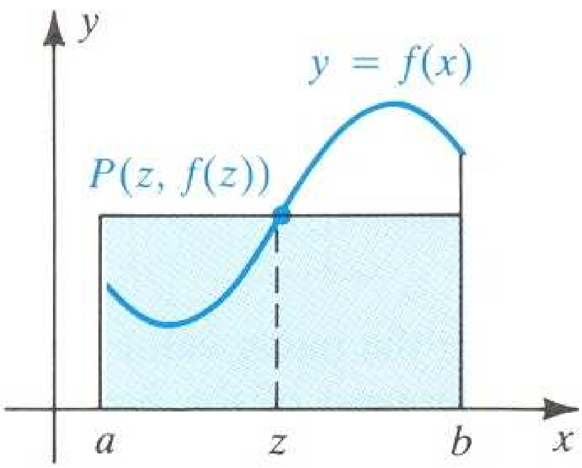
$$x^2 + 2 \geq x - 1$$

for every x *in* $[-1, 2]$ *, the conclusion follows from Corollary (5.27) .*

Mean value theorem for definite integrals (5.28) : ***Page (279)***

> ***If f is continuous on a closed interval $[a, b]$, then there is a number z in the open interval (a, b) such that***
>
> $$\int_a^b f(x)\,dx = f(z)(b - a) .$$

Figure 5.26



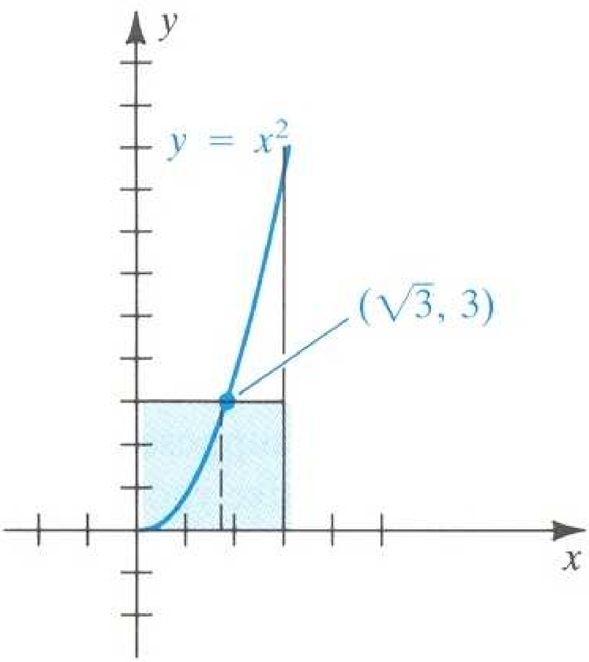
Example (5) : ***Page (280)***

It will follow from the results of Section 5.6 that $\int_0^3 x^2\,dx = 9$. Find a number z that satisfies the conclusion of the mean value theorem (5.28) for this definite integral .

Solution

*** The graphs of $f(x) = x^2$ for $0 \leq x \leq 3$ is sketched in Figure 5.27 .***

Figure 5.27



* *By the mean value theorem , there is a number* z *between* 0 *and* 3 *such that*

$$\int_0^3 x^2 \, dx = f(z)(3-0) = z^2(3)$$

* *This implies that*

$$9 = 3z^2 \,, \quad \text{or} \quad z^2 = 3$$

* *The solution of the last equation are* $z = \pm\sqrt{3}$ *; however ,* $-\sqrt{3}$ *is not in* $[0,3]$ *.*

The number $z = \sqrt{3}$ *satisfies the conclusion of the theorem .*

* *If we consider the horizontal line through* $P\left(\sqrt{3},3\right)$ *, then the area of the rectangle bounded by this line , the x-axis , and the lines* $x = 0$ *and* $x = 3$ *is equal to the area under the graph of* f *from* $x = 0$ *to* $x = 3$ *(see Figure 5.27) .*

Exercises 5.5: **5, 11, 17.**

5.6 THE FUNDAMENTAL THEOREM OF CALCULUS :

Page (282)

Fundamental theorem of calculus (5.30) : *Page (282)*

Suppose f *is continuous on a closed interval* $[a,b]$ *.*

Part I *If the function* G *is defined by*

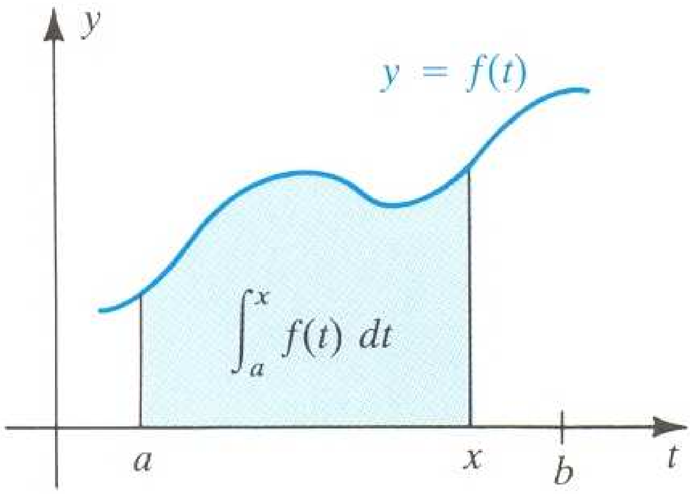
$$G(x)=\int_a^x f(t)\,dt$$

for every x in $[a,b]$, then G is an antiderivative of f on $[a,b]$.

Part II *If F is any antiderivaive of f on $[a,b]$, then*

$$\int_a^b f(x)\,dx=F(b)-F(a).$$

Figure 5.28



Corollary (5.31) : *Page (284)*

If f is continuous on $[a,b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx=F(x)\Big]_a^b=F(b)-F(a).$$

Example (1) : *Page (285)*

Evaluate $\displaystyle\int_{-2}^{3}\left(6x^2-5\right)dx$.

Solution

* An antiderivative of $6x^2 - 5$ is $F(x) = 6x^2 - 5$. Applying Corollary (5.31), we get

$$\int_{-2}^{3} \left(6x^2 - 5\right) dx = \left[2x^3 - 5x\right]_{-2}^{3}$$

Remember that :

$$* \quad \int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \left[2(3)^3 - 5(3)\right] - \left[2(-2)^3 - 5(-2)\right]$$

$$= [54 - 15] - [-16 - 10] = \boxed{35}.$$

Theorem (5.32) : Page (285)

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b.$$

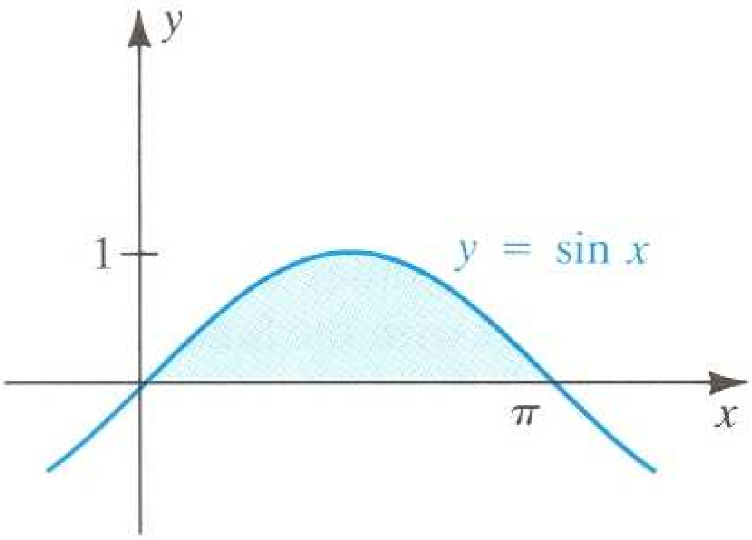
Example (2) : Page (285)

Find the area A of the region between the graph of $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$.

Solution

* The region is sketched in Figure 5.31 .

Figure 5.31



* Applying Theorem (5.19) and (5. 32) gives us the following :

$$A = \int_0^{\pi} \sin x \, dx = \left[\int \sin x \, dx \right]_0^{\pi}$$

Remember that :

$$* \quad \int \sin x \, dx = -\cos x + C$$

$$= \left[-\cos x \right]_0^{\pi}$$

$$= -\cos \pi - (-\cos 0)$$

Remember that :

$$* \quad \cos 0 = 1 \quad , \quad \cos \pi = -1$$

$$= -(-1) - (-1) = \boxed{2} .$$

Example (3) : Page (286)

Evaluate $\displaystyle\int_{-1}^{2} \left(x^3 + 1 \right)^2 \, dx$.

Solution

* We first square the integrand and then apply the power rule to each term as follows :

Remember that :

$$* \quad (a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$\int_{-1}^{2}\left(x^{3}+1\right)^{2}dx=\int_{-1}^{2}\left(x^{6}+2x^{3}+1\right)dx$$

Remember that :

$$*\ \int x^{r}\,dx=\frac{x^{r+1}}{r+1}+C\quad (r\neq -1)$$

$$=\left[\frac{x^{7}}{7}+2\,.\,\frac{x^{4}}{4}+x\right]_{-1}^{2}$$

$$=\left[\frac{2^{7}}{7}+2\,.\,\frac{2^{4}}{4}+x\right]-\left[\frac{(-1)^{7}}{7}+2\,.\,\frac{(-1)^{4}}{4}+(-1)\right]$$

$$=\boxed{\frac{405}{14}}.$$

Example (4) : *Page (286)*

Evaluate $\displaystyle\int_{1}^{4}\left(5x-2\sqrt{x}+\frac{32}{x^{3}}\right)dx$.

Solution

$$\int_{1}^{4}\left(5x-2\sqrt{x}+\frac{32}{x^{3}}\right)dx$$

** We begin by changing the form of the integrand so that the power rule may be applied to each term . thus ,*

Remember that :

$$*\ \int x^{r}\,dx=\frac{x^{r+1}}{r+1}+C\quad (r\neq -1)$$

$$\int_{1}^{4}\left(5x-2\sqrt{x}+\frac{32}{x^{3}}\right)dx=\left[5\left(\frac{x^{2}}{2}\right)-2\left(\frac{x^{3/2}}{3/2}\right)+32\left(\frac{x^{-2}}{-2}\right)\right]_{1}^{4}$$

$$=\left[\frac{5}{2}x^{2}-\frac{4}{3}x^{3/2}-\frac{16}{x^{2}}\right]_{1}^{4}$$

$$=\left[\frac{5}{2}(4)^{2}-\frac{4}{3}(4)^{3/2}-\frac{16}{4^{2}}\right]-\left[\frac{5}{2}-\frac{4}{3}-16\right]$$

$$=\boxed{\frac{259}{6}}.$$

Theorem (5.33) : ***Page (287)***

$$\boxed{\text{If } u=g(x)\text{, then } \int_{a}^{b} f(g(x))\,dx=\int_{g(a)}^{g(b)} f(u)\,du\ .}$$

Example (5) : ***Page (287)***

Evaluate $\displaystyle\int_{2}^{10}\frac{3}{\sqrt{5x-1}}\,dx$.

Solution

$$\int_{2}^{10}\frac{3}{\sqrt{5x-1}}\,dx$$

* *The integral may be written as in Formulas from (5.4) (2) by using the substitution*

$$\boxed{u=5x-1}$$

$$du=5\,dx\quad\Rightarrow\quad\frac{1}{5}du=dx$$

* *We must change limits of integration as follows :*

$$\text{at}\quad x=2\quad\rightarrow\quad u=5(2)-1=9$$

$$\text{at}\quad x=10\quad\rightarrow\quad u=5(10)-1=49$$

* *We proceed as follows :*

$$\int_2^{10} \frac{3}{\sqrt{5x-1}}\,dx = \frac{3}{5}\int_9^{49} \frac{1}{\sqrt{u}}\,du$$

$$= \frac{3}{5}\int_9^{49} u^{-1/2}\,du$$

Remember that :

$$* \quad \int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{3}{5}\left[\frac{u^{1/2}}{1/2}\right]_9^{49}$$

$$= \frac{6}{5}\left[49^{1/2} - 9^{1/2}\right] = \boxed{\frac{24}{5}}.$$

Example (6) : *Page (288)*

Evaluate $\displaystyle\int_0^{\pi/4} (1+\sin 2x)^3 \cos 2x\,dx$.

Solution

$$\int_0^{\pi/4} (1+\sin 2x)^3 \cos 2x\,dx$$

* *The integral may be written as in Formulas from (5.4) (2) by using the substitution*

$$\boxed{u = 1 - \sin 2x}$$

$$du = 2\cos 2x\,dx \quad \Rightarrow \quad \frac{1}{2}du = \cos 2x\,dx$$

* *We must change limits of integration as follows :*

$$\text{at} \quad x = 0 \quad \to \quad u = 1 - \sin 2(0) = 1$$

$$\text{at} \quad x = \pi/4 \quad \to \quad u = 1 + \sin 2(\pi/4) = 2$$

Remember that :

$$* \quad \boxed{\sin 0 = 0 \quad , \quad \sin(\pi/2) = 1}$$

*** We proceed as follows :***

$$\int_0^{\pi/4} (1+\sin 2x)^3 \cos 2x \, dx = \frac{1}{2}\int_1^{\pi/4} u^3 \, dx$$

Remember that :

$$* \quad \boxed{\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)}$$

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2$$

$$= \frac{1}{8}\left[2^4 - 1^4\right] = \boxed{\frac{15}{8}}.$$

Exercises 5.6: 3, 11, 21, 27, 35.

CHAPTER (7)

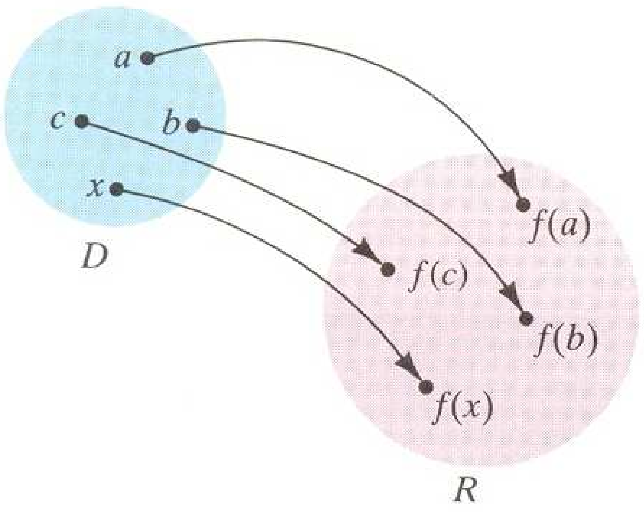
LOGARITHMIC AND EXPONENTIAL FUNCTIONS

7.1 INVERSE FUNCTIONS : Page (374)

Definition (7.1) : Page (374)

A function f with domain D and range R is a one-to-one function if whenever $a \neq b$ in D, the $f(a) \neq f(b)$ in R.

Figure 7.1

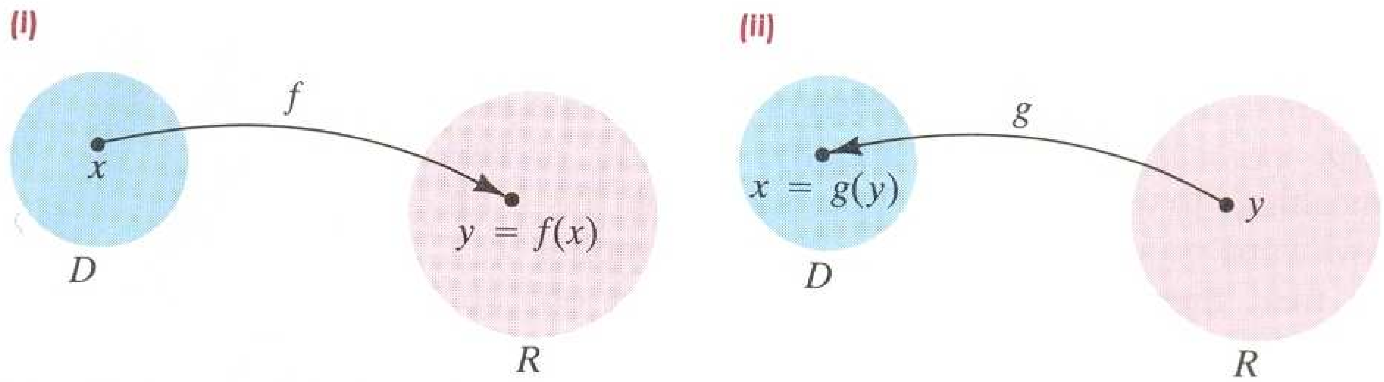


Definition (7.2) : Page (374)

Let f be one-to-one function with domain D and range R. A function g with domain R and range D is the inverse function of f, provided the following condition is true for every x in D and every y in R:

$$y = f(x) \quad \text{if and only if} \quad x = g(y).$$

Figure 7.3



Definition (7.3) : Page (375)

Let f be a one-to-one function with domain D and range R. If g is a function with domain R and range D, then g is the inverse function of f if and only if both of the following conditions are true :

(i) $g\left(f(x)\right)=x$ for every x in D.

(ii) $f\left(g(y)\right)=y$ for every y in R.

Note : Page (375)

* If a function f has an inverse function g, we often denote g by f^{-1} .

Domain and ranges of f and f^{-1} (7.4) : Page (375)

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

Guidelines for finding f^{-1} in simple cases (7.5) :

Page (376)

1 Verify that f is a one-to-one function (or that f is increasing or is decreasing) throughout its domain .

2 Solve the equation $y=f(x)$ for every x in terms of y, obtaining an equation of the form $x=f^{-1}(y)$.

3 Verify the following two conditions

$$f^{-1}\left(f(x)\right)=x \quad \text{and} \quad f\left(f^{-1}(x)\right)=x$$

for every x in the domain of f and f^{-1} , respectively .

Example (1) : Page (376)

Let $f(x)=3x-5$. Find the inverse function of f .

Solution

$$f(x)=3x-5$$

* *We shall follow the three guidelines . First , we note that the graph of the linear function f is a line of slope 3 . Since f is increasing throughout $\mathbb{R}$, f is one-to-one function and hence the inverse function f^{-1} exists . Moreover , since the domain and range of f are $\mathbb{R}$, the same is true for f^{-1} .*

* *As in guideline 2 , we consider the equation*

$$y=3x-5$$

and then solve for x in terms of y , obtaining

$$x=\frac{y+5}{3}$$

We now let

$$f^{-1}(y)=\frac{y+5}{3}$$

Since the symbol used for the variable is immaterial , we may also write

$$f^{-1}(x)=\frac{x+5}{3}$$

* *We next verify the condition (i) $f^{-1}\left(f(x)\right)=x$ and (ii) $f\left(f^{-1}(x)\right)=x$:*

(i) $f^{-1}\left(f(x)\right)=f^{-1}(3x-5)$ $\quad$ (definition of f)

$$=\frac{(3x-5)+5}{3} \qquad \text{(definition of } f^{-1}\text{)}$$

$$=x \qquad \text{(simplify)}$$

(ii) $f\left(f^{-1}(x)\right)=f\left(\dfrac{x+5}{3}\right)$ $\quad$ (definition of f^{-1})

$$= 3\left(\frac{x+5}{3}\right) \qquad (\textit{definition of } f\,)$$

$$= x \qquad (\textit{simplify}\,)$$

* *Thus , by Theorem (7.3) , the inverse function of f is given by*

$$\boxed{f^{-1}(x) = \frac{x+5}{3}}\,.$$

Example (2) : *Page (377)*

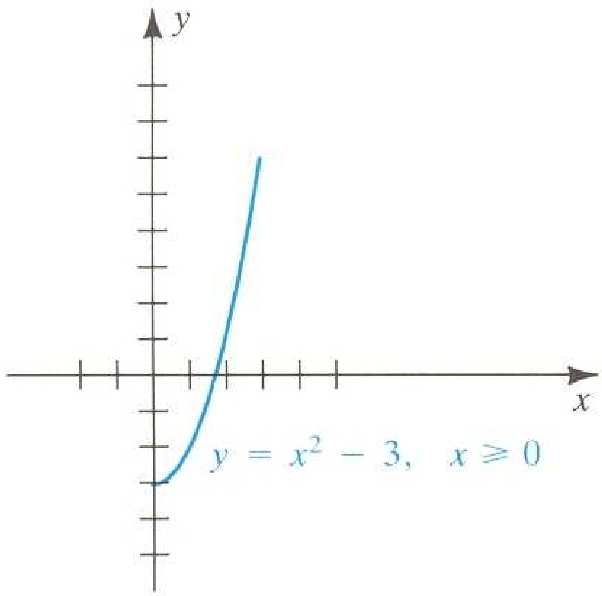
Let $f(x) = x^2 - 3$ *for* $x \geq 0$. *Find the inverse function of f .*

Solution

$$f(x) = x^2 - 3 \quad \text{for} \quad x \geq 0$$

* *The graph of f is sketched in Figure 7.4 .*

Figure 7.4



* *The domain of f is $[0, \infty)$, and the range is $[-3, \infty)$. Since f is increasing , it is one-to-one function and hence has a inverse function f^{-1} that has domain $[-3, \infty)$ and range $[0, \infty)$.*

* *As in guideline 2 , we consider the equation*

$$y = x^2 - 3$$

and then solve for x in terms of y, obtaining

$$x = \pm\sqrt{y+3}$$

Since x is nonnegative, we reject $x = -\sqrt{y+3}$ and let

$$f^{-1}(y) = \sqrt{y+3}\ , \quad \text{or, equivalently,} \quad f^{-1}(x) = \sqrt{x+3}\ .$$

* *Finally, we verify that*

(i) $f^{-1}(f(x)) = x$ *for* x *in* $[0, \infty)$ *and*

(ii) $f(f^{-1}(x)) = x$ *for* x *in* $[-3, \infty)$:

(i) $$f^{-1}(f(x)) = f^{-1}(x^2 - 3)$$
$$= \sqrt{(x^2 - 3) + 3}$$
$$= \sqrt{x^2} = x \quad \text{if} \quad x \ge 0$$

(ii) $$f(f^{-1}(x)) = f\left(\sqrt{x+3}\right)$$
$$= \left(\sqrt{x+3}\right)^2 - 3$$
$$= (x+3) - 3 = x \quad \text{if} \quad x \ge -3$$

* *Thus, by Theorem (7.3), the inverse function of f is given by*

$$\boxed{f^{-1}(x) = \sqrt{x+3} \quad \text{for} \quad x \ge -3}\ .$$

Exercise 7.1:No. **1, 3, 9.**

7.2 THE NATURAL LOGARITHMIC FUNCTION :

Page (381)

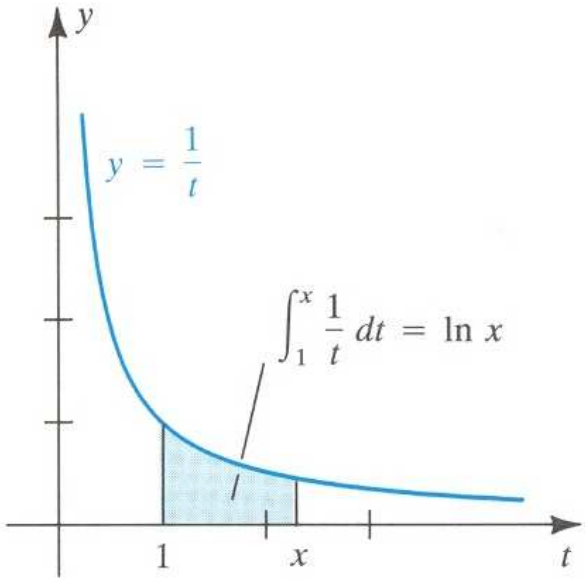
Definition (7.9) : *Page (382)*

The natural logarithmic function, denoted by ln, is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

for every $x > 0$.

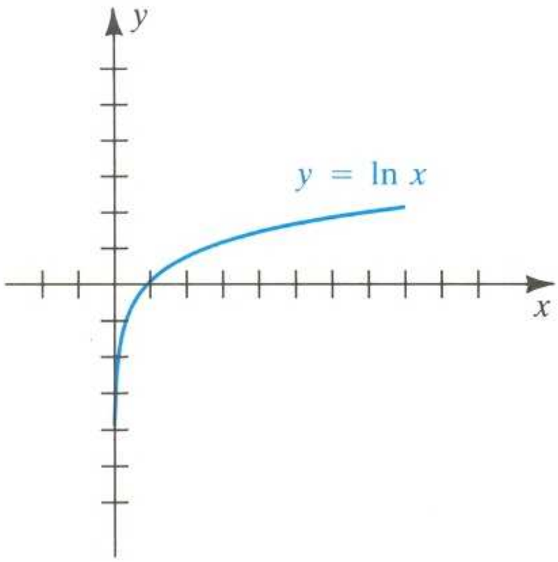
Figure 7.8



Theorem (7.10) : *Page (383)*

$$D_x \ln x = \frac{1}{x} .$$

Figure 7.9



Theorem (7.11) : *Page (384)*

If $u = g(x)$ *and* g *is differentiable , then*

(i) $$D_x \ln u = \frac{1}{u} D_x u \quad \text{if} \quad g(x) > 0 .$$

(ii) $$D_x \ln |u| = \frac{1}{u} D_x u \quad \text{if} \quad g(x) \neq 0 .$$

Example (1) : *Page (385)*

If $f(x) = \ln(x^2 - 6)$ *, find* $f'(x)$.

Solution

$$f(x) = \ln(x^2 - 6)$$

* *Applying Theorem (7.11) (i) , with* $u = x^2 - 6$ *, we obtain*

Remember that :

$$* \quad D_x \ln u = \frac{1}{u} D_x u$$

$$f'(x) = D_x \ln(x^2 - 6)$$

$$= \frac{1}{x^2-6} D_x\left(x^2-6\right)$$

$$= \boxed{\frac{2x}{x^2-6}}.$$

<u>**Example (2) :**</u> ***Page (385)***

If $y = \ln\sqrt{x+1}$ *, find* $\dfrac{dy}{dx}$.

Solution

$$y = \ln\sqrt{x+1}$$

* *Applying Theorem (7.11) (i) , with* $u = \sqrt{x+1}$ *, we obtain*

<u>***Remember that :***</u>

$$* \quad \boxed{D_x \ln u = \frac{1}{u} D_x u}$$

$$\frac{dy}{dx} = \frac{d}{dx} \ln\sqrt{x+1}$$

$$= \frac{1}{\sqrt{x+1}} \frac{d}{dx}\sqrt{x+1} = \frac{1}{\sqrt{x+1}} \cdot \frac{1}{2}(x+1)^{-1/2}$$

$$= \frac{1}{\sqrt{x+1}} \cdot \frac{1}{2} \frac{1}{\sqrt{x+1}} = \boxed{\frac{1}{2(x+1)}}.$$

<u>**Theorem (7.12) :**</u> ***Page (385)***

If $p > 0$ *and* $q > 0$ *, then*

(i) $\ln pq = \ln p + \ln q$.

(ii) $\ln \dfrac{p}{q} = \ln p - \ln q$.

(iii) $\ln p^{r} = r \ln p$ for every rational number r.

ILLUSTRATION : Page (387)

$f(x)$	$f(x)$ after using laws of logarithms	$f'(x)$
$\ln\left[(x+2)(3x-5)\right]$	$\ln(x+2)+\ln(3x-5)$	$\frac{1}{x+2}+\frac{1}{3x-5}\cdot 3 = \boxed{\frac{6x+1}{(x+2)(3x-5)}}$.
$\ln\frac{x+2}{3x-5}$	$\ln(x+2)-\ln(3x-5)$	$\frac{1}{x+2}-\frac{1}{3x-5}\cdot 3 = \boxed{\frac{-11}{(x+2)(3x-5)}}$.
$\ln\left(x^2+1\right)^5$	$5\ln\left(x^2+1\right)$	$5\cdot\frac{1}{x^2+1}\cdot 2x = \boxed{\frac{10x}{x^2+1}}$.
$\ln\sqrt{x+1}$	$\frac{1}{2}\ln(x+1)$	$\frac{1}{2}\cdot\frac{1}{x+1} = \boxed{\frac{1}{2(x+1)}}$.

Example (4) : Page (387)

If $f(x)=\ln\left[\sqrt{6x-1}\,(4x+5)^3\right]$, find $f'(x)$.

Solution

$$f(x)=\ln\left[\sqrt{6x-1}\,(4x+5)^3\right]$$

* We first write $\sqrt{6x+1}=(6x+1)^{1/2}$ and the use laws of logarithms (i) and (iii) :

$$f(x)=\ln\left[(6x-1)^{1/2}\,(4x+5)^3\right]$$

$$= \ln (6x-1)^{1/2} + \ln (4x+5)^3$$

Remember that :

$$* \quad \ln p^r = r \ln p \quad , \quad \ln (pq) = \ln p + \ln q$$

$$= \frac{1}{2} \ln (6x-1) + 3 \ln (4x+5)$$

* By Theorem (7.11) ,

Remember that :

$$* \quad D_x \ln u = \frac{1}{u} D_x u$$

$$f'(x) = \left(\frac{1}{2} \cdot \frac{1}{6x-1} \cdot 6 \right) + \left(3 \cdot \frac{1}{4x+5} \cdot 4 \right)$$

$$= \frac{3}{6x-1} + \frac{12}{4x+5}$$

$$= \boxed{\frac{84x+3}{(6x-1)(4x+5)}} .$$

Example (5) : Page (387)

If $y = \ln \sqrt[3]{\frac{x^2-1}{x^2+1}}$, find $\frac{dy}{dx}$.

Solution

$$y = \ln \sqrt[3]{\frac{x^2-1}{x^2+1}}$$

* We first use laws of logarithms to change the form of y as follows :

$$y = \ln \left(\frac{x^2-1}{x^2+1} \right)^{1/3}$$

$$=\frac{1}{3}\ln\left(\frac{x^2-1}{x^2+1}\right)$$

Remember that :

$$* \quad \ln p^r = r\ \ln p \quad , \quad \ln\left(\frac{p}{q}\right)=\ln p-\ln q$$

$$=\frac{1}{3}\left[\ln\left(x^2-1\right)-\ln\left(x^2+1\right)\right]$$

*** Next we use Theorem (7.11) to obtain ,***

Remember that :

$$* \quad D_x \ln u=\frac{1}{u}D_x u$$

$$\frac{dy}{dx}=\frac{1}{3}\left(\frac{1}{x^2-1}.2x-\frac{1}{x^2+1}.2x\right)$$

$$=\frac{2x}{3}\left(\frac{1}{x^2-1}-\frac{1}{x^2+1}\right)$$

$$=\frac{2x}{3}\left[\frac{2}{\left(x^2-1\right)\left(x^2+1\right)}\right]$$

$$=\boxed{\frac{4x}{3\left(x^2-1\right)\left(x^2+1\right)}}.$$

Guidelines for logarithmic differentiation (7.13) : ***Page (376)***

1 $y=f(x)$	*(given)*
2 $\ln y=\ln f(x)$	*(take natural logarithms and simplify)*
3 $D_x\left[\ln y\right]=D_x\left[\ln f(x)\right]$	*(differentiate implicitly)* *(by Theorem (7.11))*

$$4 \quad \frac{1}{y} D_x y = D_x \left[\ln f(x) \right] \qquad \text{(multiply by } y = f(x)\text{)}$$

$$5 \quad D_x y = f(x) D_x \left[\ln f(x) \right]$$

Example (6) : *Page (388)*

If $y = \dfrac{(5x-4)^3}{\sqrt{2x+1}}$ *, use logarithmic differentiation to find* $D_x y$ *.*

Solution

$$y = \frac{(5x-4)^3}{\sqrt{2x+1}}$$

** As in guideline 2 , we begin by taking the natural logarithm of each side , obtaining*

$$\ln y = \ln \frac{(5x-4)^3}{\sqrt{2x+1}}$$

$$= \ln (5x-4)^3 - \ln \sqrt{2x+1}$$

Remember that :

$$* \quad \ln p^r = r \ln p \quad , \quad \ln \frac{p}{q} = \ln p - \ln q$$

$$= 3 \ln (5x-4) - \frac{1}{2} \ln (2x+1)$$

** Applying guidelines 3 and 4 , we differentiate implicitly with respect to* x *and the use Theorem (7.8) to obtain*

Remember that :

$$* \quad D_x \ln u = \frac{1}{u} D_x u$$

$$\frac{1}{y} D_x y = \left(3 \cdot \frac{1}{5x-4} \cdot 5 \right) - \left(\frac{1}{2} \cdot \frac{1}{2x+1} \cdot 2 \right)$$

$$= \frac{25x+19}{(5x-4)(2x+1)}$$

* *Finally , as in guideline 5 , we multiply both sides of the last equation by y (that is , by $(5x-4)^3 / \sqrt{2x+1}$) to get*

$$D_x y = \frac{25x+19}{(5x-4)(2x+1)} \cdot \frac{(5x-4)^3}{\sqrt{2x+1}}$$

$$= \boxed{\frac{(25x+19)(5x-4)^2}{(2x+1)^{3/2}}}.$$

Exercise 7.2:No. **1, 3, 7, 15, 21, 23, 25, 31.**

7.3 THE NATURAL EXPONENTIAL FUNCTION :

Page (392)

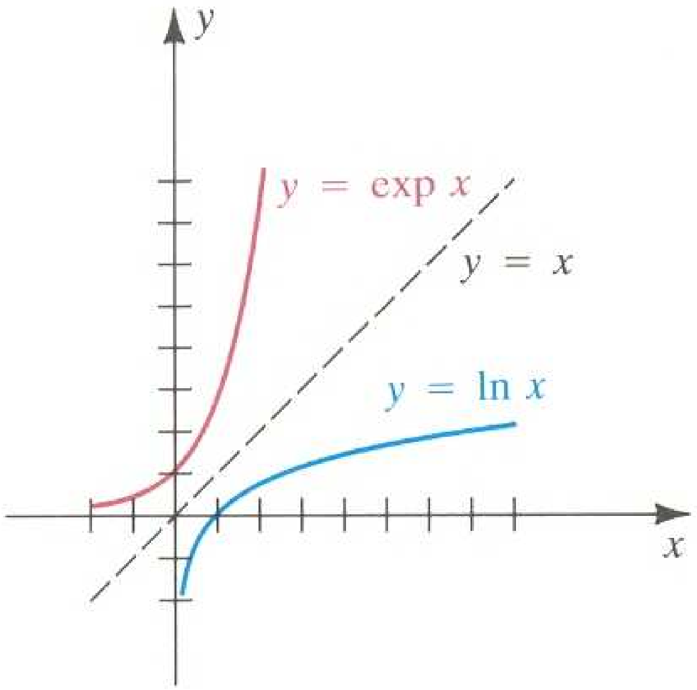
Theorem (7.14) : *Page (392)*

To every real number x there corresponds exactly one positive real number y such that $\ln y = x$.

Theorem (7.15) : *Page (392)*

The natural exponential function , denoted by $\exp x$ or e^x , is the inverse of the natural logarithmic function $\ln x$.

Figure 7.12



Definition (7.16) : *Page (393)*

The letter e denotes the positive real number such that $\ln e = 1$.

Approximation to e (7.17) : *Page (393)*

$$e \approx 2.71828 .$$

Definition of e^x (7.18) : *Page (393)*

If x is any real number , then

$$e^x = y \quad \text{if and only if} \quad \ln y = x .$$

Theorem (7.19) : *Page (394)*

(i) $\ln e^x = x$ for every x .

(ii) $e^{\ln x} = x$ for every $x > 0$.

ILLUSTRATION : *Page (394)*

* $\ln e^{5} = \boxed{5}$.

* $\ln e^{\sqrt{x+1}} = \boxed{\sqrt{x+1}}$.

* $e^{\ln 5} = \boxed{5}$.

* $e^{\ln\sqrt{x+1}} = \boxed{\sqrt{x+1}}$.

* $e^{3\ln x} = \left(e^{\ln x}\right)^{3} = \boxed{x^{3}}$.

* $e^{k\ln x} = \left(e^{\ln x}\right)^{k} = \boxed{x^{k}}$.

Theorem (7.20) : *Page (394)*

If p and q are real numbers and r is a rational number , then

(i) $e^{p}e^{q} = e^{p+q}$. (ii) $\dfrac{e^{p}}{e^{q}} = e^{p-q}$. (iii) $\left(e^{p}\right)^{r} = e^{pr}$.

Theorem (7.21) : *Page (394)*

$$D_x e^{x} = e^{x}.$$

Example (1) : *Page (395)*

If $f(x) = x^{2}e^{x}$ *, find* $f'(x)$.

Solution

$$f(x) = x^{2}e^{x}$$

* *By the product rule and Theorem (7.21) ,*

Remember that :

* $D_x\left[f(x)g(x)\right] = f(x)\,D_x g(x) + g(x)\,D_x f(x)$

$$* \quad \boxed{D_x \ln u = \frac{1}{u} D_x u}$$

$$f'(x) = x^2 \left(D_x e^x\right) + e^x \left(D_x x^2\right)$$

$$= x^2 e^x + e^x (2x)$$

$$= \boxed{x e^x (x+2)}.$$

Theorem (7.22) : *Page (395)*

If $u = g(x)$ and g is differentiable, then

$$D_x e^u = e^u D_x u \; .$$

Example (2) : *Page (395)*

If $y = e^{\sqrt{x^2+1}}$, find dy/dx .

Solution

$$y = e^{\sqrt{x^2+1}}$$

* By Theorem (7.22) ,

Remember that :

$$* \quad \boxed{D_x e^u = e^u \; D_x u}$$

$$\frac{dy}{dx} = \frac{d}{dx} e^{\sqrt{x^2+1}}$$

$$= e^{\sqrt{x^2+1}} \frac{d}{dx} \left(x^2+1\right)^{1/2}$$

$$= e^{\sqrt{x^2+1}} \left(\frac{1}{2}\right) \left(x^2+1\right)^{-1/2} (2x)$$

$$= e^{\sqrt{x^2+1}} \cdot \frac{x}{\sqrt{x^2+1}}$$

$$= \boxed{\frac{x\,e^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}}.$$

Exercise 7.3:No. **1,3,5,7,9,11.**

7.4 INTEGRATION : ***Page (399)***

Theorem (7.23) : ***Page (400)***

If $u = g(x) \neq 0$ ***and*** g ***is differentiable , then***

$$\int \frac{1}{u}\,du = \ln|u| + C \ .$$

Remember that :

$*$ ***Since*** $D_x \ln|g(x)| = \frac{1}{g(x)}\, g'(x)$ ***, then***

$$\int \frac{1}{g(x)}\, g'(x)\,dx = \ln|g(x)| + C$$

Example (1) : ***Page (400)***

Evaluate $\int \frac{x}{3x^2-5}\,dx$.

Solution

$$\int \frac{x}{3x^2-5}\,dx$$

** The integral may be written as in Theorem (7.23) by using the substitution*

$$\boxed{u = 3x^2 - 5}$$

$$du = 6x\,dx \quad \Rightarrow \quad \frac{1}{6}du = x\,dx$$

* *We proceed as follows :*

$$\int \frac{x}{3x^2 - 5}\,dx = \frac{1}{6}\int \frac{1}{u}\,du$$

Remember that :

* $\boxed{\int \frac{1}{u}\,du = \ln|u| + C}$

$$= \frac{1}{6}\ln|u| + C$$

$$= \boxed{\frac{1}{6}\ln\left|3x^2 - 5\right| + C}\,.$$

Example (2) : *Page (400)*

Evaluate $\displaystyle\int_2^4 \frac{1}{9 - 2x}\,dx\,.$

Solution

$$\int_2^4 \frac{1}{9 - 2x}\,dx$$

* *The integral may be written as in Theorem (7.23) by using the substitution*

$$\boxed{u = 9 - 2x}$$

$$du = -2\,dx \quad \Rightarrow \quad -\frac{1}{2}du = dx$$

* *We must change limits of integration as follows :*

$$\text{at} \quad x = 2 \quad \rightarrow \quad u = 9 - 2(2) = 5$$

$$at \quad x = 4 \quad \rightarrow \quad u = 9 - 2(4) = 1$$

** We proceed as follows :*

$$\int_2^4 \frac{1}{9-2x}\,dx = -\frac{1}{2}\int_5^1 \frac{1}{u}\,du$$

Remember that :

$$* \; \int \frac{1}{u}\,du = \ln|u| + C$$

$$= -\frac{1}{2}\Big[\ln|u|\Big]_5^1$$

$$= -\frac{1}{2}\Big[\ln|1| - \ln|5|\Big]$$

$$= \frac{1}{2}\Big[-\ln|1| + \ln|5|\Big]$$

Remember that :

$$* \; \ln\left(\frac{p}{q}\right) = \ln p - \ln q$$

$$= \boxed{\frac{1}{2}\ln 5}.$$

Another metod :

** We first find the indefinite integral as before*

$$\int \frac{1}{9-2x}\,dx = -\frac{1}{2}\ln|9-2x| + C$$

** Applying the fundamental theorem of calculus yields*

$$\int_2^4 \frac{1}{9-2x}\,dx = -\frac{1}{2}\Big[\ln|9-2x|\Big]_2^4$$

$$= -\frac{1}{2}(\ln 1 - \ln 5)$$

$$= \boxed{\frac{1}{2}\ln 5}.$$

Example (3) : *Page (401)*

Evaluate $\displaystyle\int \frac{\sqrt{\ln x}}{x}\,dx$.

Solution

$$\int \frac{\sqrt{\ln x}}{x}\,dx$$

* *The integral may be written as in Theorem (5.4) by using the substitution*

$$\boxed{u = \ln x}$$

$$du = \frac{1}{x}\,dx$$

* *We proceed as follows :*

$$\int \frac{\sqrt{\ln x}}{x}\,dx = \int u^{1/2}\,du$$

Remember that :

$$* \quad \int u^r\,dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{u^{3/2}}{(3/2)} + C$$

$$= \boxed{\frac{2}{3}(\ln x)^{3/2} + C}.$$

Theorem (7.24) : Page (401)

If $u = g(x)$ and g is differentiable, then

$$\int e^u \, du = e^u + C .$$

Remember that :

$$* \quad \text{Since} \quad D_x e^{g(x)} = e^{g(x)} \, g'(x), \text{ then}$$
$$\int e^{g(x)} \, g'(x) \, dx = e^{g(x)} + C$$

Note : Page (402)

* As a special case of Theorem (7.24), if $u = x$, then

$$\boxed{\int e^x \, dx = e^x + C}$$

Example (4) : Page (402)

Evaluate

(a) $\displaystyle\int \frac{e^{3/x}}{x^2} \, dx$.

(b) $\displaystyle\int_1^2 \frac{e^{3/x}}{x^2} \, dx$.

Solution

$$(a) \int \frac{e^{3/x}}{x^2} \, dx$$

* The integral may be written as in Theorem (7.24) by using the substitution

$$\boxed{u = \frac{3}{x}}$$

$$du = -\frac{3}{x^2} dx \quad \Rightarrow \quad -\frac{1}{3} du = \frac{1}{x^2} dx$$

* We proceed as follows :

$$\int \frac{e^{3/x}}{x^2}\,dx = -\frac{1}{3}\int e^u\,du$$

Remember that :

$$* \ \boxed{\int e^u\,du = e^u + C}$$

$$= -\frac{1}{3}e^u + C$$

$$= \boxed{-\frac{1}{3}e^{3/x} + C}.$$

$$(b)\ \int_1^2 \frac{e^{3/x}}{x^2}\,dx$$

* The integral may be written as in Theorem (7.24) by using the substitution

$$\boxed{u = \frac{3}{x}}$$

$$du = -\frac{3}{x^2}dx \quad \Rightarrow \quad -\frac{1}{3}du = \frac{1}{x^2}dx$$

* We must change limits of integration as follows :

$$\text{at} \quad x = 1 \quad \rightarrow \quad u = \frac{3}{1} = 3$$

$$\text{at} \quad x = 2 \quad \rightarrow \quad u = \frac{3}{2} = \frac{3}{2}$$

* We proceed as follows :

$$\int_1^2 \frac{e^{3/x}}{x^2}\,dx = -\frac{1}{3}\int_3^{3/2} e^u\,du$$

Remember that :

$$* \ \boxed{\int e^u\,du = e^u + C}$$

$$= -\frac{1}{3}\left[e^{u}\right]_{3}^{3/2}$$

$$= \boxed{-\frac{1}{3}\left(e^{3/2} - e^{3}\right) \approx 5.2}\,.$$

Note : Page (402)

* We can show that

$$\boxed{\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C \quad , \quad a \neq 0}$$

ILLUSTRATION : Page (403)

* $\int e^{3x}\,dx = \boxed{\frac{1}{3}e^{3x} + C}$.

* $\int e^{-x}\,dx = \boxed{-e^{-x} + C}$.

* $\int e^{-5x}\,dx = \boxed{-\frac{1}{5}e^{-5x} + C}$.

Theorem (7.25) : Page (404)

(i) $\int \tan u\,du = -\ln|\cos u| + C$.

(ii) $\int \cot u\,du = \ln|\sin u| + C$.

(iii) $\int \sec u\,du = \ln|\sec u + \tan u| + C$.

(iv) $\int \csc u\,du = \ln|\csc u + \cot u| + C$.

Example (7) : Page (405)

Evaluate $\int x \cot x^{2}\,dx$.

Solution

$$\int x \cot x^2 \, dx$$

* The integral may be written as in Theorem (7.25) (ii) by using the substitution

$$\boxed{u = x^2}$$

$$du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}du = x\,dx$$

* We proceed as follows :

$$\int x \cot x^2 \, dx = \frac{1}{2} \int \cot u \, du$$

Remember that :

$$* \quad \boxed{\int \cot u \, du = \ln \left| \sin u \right| + C}$$

$$= \frac{1}{2} \ln \left| \sin u \right| + C$$

$$= \boxed{\frac{1}{2} \ln \left| \sin x^2 \right| + C} \, .$$

Example (8) : Page (405)

Evaluate $\displaystyle\int_0^{\pi/2} \tan \frac{x}{2} \, dx$.

Solution

$$\int_0^{\pi/2} \tan \frac{x}{2} \, dx$$

* The integral may be written as in Theorem (7.25) (i) by using the substitution

$$\boxed{u = \frac{x}{2}}$$

$$du = \frac{1}{2}dx \quad \Rightarrow \quad 2du = dx$$

* We must change limits of integration as follows :

$$\text{at} \quad x = 0 \quad \to \quad u = \frac{0}{2} = 0$$

$$\text{at} \quad x = \pi/2 \quad \to \quad u = \frac{\pi/2}{2} = \frac{\pi}{4}$$

* We proceed as follows :

$$\int_0^{\pi/2} \tan\frac{x}{2}\, dx = 2\int_0^{\pi/4} \tan u\, du$$

Remember that :

$$* \quad \int \tan u\, du = -\ln\left|\cos u\right| + C$$

$$= -2\left[\ln\left|\cos u\right|\right]_0^{\pi/4}$$

$$= -2\left[\ln\left|\cos\frac{\pi}{4}\right| - \ln\left|\cos 0\right|\right]$$

Remember that :

$$* \quad \cos 0 = 1 \;,\; \cos\pi/4 = \frac{1}{\sqrt{2}} \;,\; \ln 1 = 0$$

$$= -2\left[\ln\left|\frac{1}{\sqrt{2}}\right| - \ln\left|1\right|\right]$$

$$= 2\left[-\ln\left|\frac{1}{\sqrt{2}}\right| + 0\right]$$

$$= \ln\left(\frac{1}{\sqrt{2}}\right)^{-2} = \ln\left(\sqrt{2}\right)^2$$

Remember that :

$$* \ \boxed{\ln p^r = r \ln p}$$

$$= \boxed{\ln 2 \approx 0.69}\ .$$

Example (9) : *Page (406)*

Evaluate $\int x \cot x^2 \, dx$.

Solution

$$\int e^{2x} \sec e^{2x} \, dx$$

* *The integral may be written as in Theorem (7.25) (iii) by using the substitution*

$$\boxed{u = e^{2x}}$$

$$du = 2 e^{2x} dx \quad \Rightarrow \quad \frac{1}{2} du = e^{2x} dx$$

* *We proceed as follows :*

$$\int e^{2x} \sec e^{2x} \, dx = \frac{1}{2} \int \sec u \, du$$

Remember that :

$$* \ \boxed{\int \sec u \, du = \ln \left| \sec u + \tan u \right| + C}$$

$$= \frac{1}{2} \ln \left| \sec u + \tan u \right| + C$$

$$= \boxed{\frac{1}{2} \ln \left| \sec e^{2x} + \tan e^{2x} \right| + C}\ .$$

Example (10) : *Page (406)*

Evaluate $\int (\csc x - 1)^2 \, dx$.

Solution

$$\int (\csc x - 1)^2\,dx = \int \left(\csc^2 x - 2\csc x + 1\right) dx$$

Remember that :

$$* \quad (a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$= \int \csc^2 x\,dx - 2\int \csc x\,dx + \int dx$$

Remember that :

$$* \quad \int \csc^2 u\,du = -\cot u + C$$

$$* \quad \int \csc u\,du = \ln\left|\csc u - \cot u\right| + C$$

$$* \quad \int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \boxed{-\cot x - 2\ln\left|\csc x - \cot x\right| + x + C}\,.$$

Exercise 7.4:No. **1(a), 3(a), 5(a), 7(a), 15.**

7.5 GENERAL EXPONENTIAL AND LOGARITHMIC FUNCTIONS : Page (408)

Definition of a^x (7.23) : *Page (408)*

$$a^x = e^{x\ln a}\,.$$

For every $a > 0$ *and every real number* x .

Remember that :

$$* \quad a^x = e^{\ln a^x} = e^{x\ln a}$$

ILLUSTRATION : *Page (408)*

$$* \quad 2^\pi = e^{\pi \ln 2} \approx \boxed{e^{2.18} \approx 8.8}\,.$$

* $\left(\frac{1}{2}\right)^{\sqrt{3}} = e^{\sqrt{3}\ln(1/2)} \approx \boxed{e^{-1.20} \approx 0.3}$.

Laws of exponents (7.27) : *Page (408)*

Let $a > 0$ and $b > 0$. If u and v are any real numbers, then

$$a^u a^v = a^{u+v} \quad , \quad \left(a^u\right)^v = a^{uv} \quad , \quad (ab)^u = a^u b^u \, ,$$

$$\frac{a^u}{a^v} = a^{u+v} \quad , \quad \left(\frac{a}{b}\right)^u = \frac{a^u}{b^u} \, .$$

Theorem (7.28) : *Page (409)*

(i) $D_x a^x = a^x \ln a$. (ii) $D_x a^u = \left(a^u \ln a\right) D_x u$.

ILLUSTRATION : *Page (409)*

* $D_x 3^x = \boxed{3^x \ln 3}$.

* $D_x 10^x = \boxed{10^x \ln 10}$.

* $D_x 3^{\sqrt{x}} = \left(3^{\sqrt{x}} \ln 3\right) D_x \sqrt{x}$

$$= \left(3^{\sqrt{x}} \ln 3\right)\left(\frac{1}{2\sqrt{x}}\right) = \boxed{\frac{3^{\sqrt{x}} \ln 3}{2\sqrt{x}}} \, .$$

* $D_x 10^{\sin x} = \left(10^{\sin x} \ln 10\right) D_x \sin x = \boxed{\left(10^{\sin x} \ln 10\right) \cos x}$.

Example (1) : *Page (410)*

Find y' if $y=\left(x^2+1\right)^{10}+10^{x^2+1}$.

Solution

$$y=\left(x^2+1\right)^{10}+10^{x^2+1}$$

* Using the power rule for functions and Theorem (7.28), we obtain

Remember that :

$$* \; D_x u^r = r\, u^{r-1}\; D_x u$$

$$* \; D_x a^u = a^u \; D_x u$$

$$y'=10\left(x^2+1\right)^9(2x)+\left(10^{x^2+1}\ln 10\right)(2x)$$

$$=\boxed{20x\left[\left(x^2+1\right)^9+10^{x^2}\ln 10\right]}.$$

Theorem (7.29) : Page (410)

$$\text{(i)} \int a^x\,dx=\left(\frac{1}{\ln a}\right)a^x+C. \qquad \text{(ii)} \int a^u\,du=\left(\frac{1}{\ln a}\right)a^u+C.$$

Example (2) : Page (410)

Evaluate

(a) $\int 3^x\,dx$.

(b) $\int x\,3^{x^2}\,dx$.

Solution

(a) Using (i) of Theorem (7.29) yields

$$\int 3^x\,dx=\boxed{\left(\frac{1}{\ln 3}\right)3^x+C}.$$

Remember that :

$$* \quad \int a^{x}\,dx = \left(\frac{1}{\ln a}\right)a^{x} + C$$

(b) To Use (ii) of Theorem (7.29) , we make the substitution

$$\boxed{u = x^{2}}$$

$$du = 2\,x\,dx \quad \Rightarrow \quad \frac{1}{2}du = x\,dx$$

and proceed as follows :

$$\int x\;3^{x^{2}}\;dx = \frac{1}{2}\int 3^{u}\;du$$

Remember that :

$$* \quad \int a^{u}\,du = \left(\frac{1}{\ln a}\right)a^{u} + C$$

$$= \frac{1}{2}\left(\frac{1}{\ln 3}\right)3^{u} + C$$

$$= \boxed{\left(\frac{1}{2\,\ln 3}\right)3^{x^{2}} + C}\,.$$

Definition of $\log_{a} x$ (7.30) : Page (411)

$$\boxed{y = \log_{a} x \quad \text{if and only if} \quad x = a^{y}\,.}$$

Note : Page (411)

*** The relationship between $\log_{a}$ and $\ln$ is**

$$\boxed{\log_{a} x = \frac{\ln x}{\ln a}}$$

Theorem (7.31) : Page (412)

$$\text{(i)}\ D_x \log_a x = D_x\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\cdot\frac{1}{x}\,.$$

$$\text{(ii)}\ D_x \log_a |u| = D_x\left(\frac{\ln |u|}{\ln a}\right) = \frac{1}{\ln a}\cdot\frac{1}{u} D_x u\,.$$

ILLUSTRATION : ***Page (412)***

* $$D_x \log_2 x = D_x\left(\frac{\ln x}{\ln 2}\right) = \frac{1}{\ln 2}\cdot\frac{1}{x} = \boxed{\frac{1}{(\ln 2)x}}\,.$$

* $$D_x \log_2 |x^2-9| = D_x\left(\frac{\ln |x^2-9|}{\ln 2}\right)$$

$$= \frac{1}{\ln 2}\cdot\frac{1}{x^2-9}\cdot 2x = \boxed{\frac{2x}{(\ln 2)(x^2-9)}}\,.$$

Note : ***Page (412)***

* ***If*** $a = 10$ ***, we obtain common logarithm***

$$\boxed{\log_{10} x = \log x}\,.$$

* ***If*** $a = e \approx 2.71828$ ***, we obtain natural logarithm***

$$\boxed{\log_e x = \ln x}\,.$$

Example (4) : ***Page (412)***

If $f(x) = \log \sqrt[3]{(2x+5)^2}$ ***, find*** $f'(x)$.

Solution

$$f(x) = \log \sqrt[3]{(2x+5)^2}$$

* We first write $f(x) = \log(2x+5)^{2/3}$. The law $\log u^r = r\log u$ is true only if $u>0$; however , since $(2x+5)^{2/3} = |2x+5|^{2/3}$, we may proceed as follows :

$$f(x) = \log(2x+5)^{2/3}$$

$$= \log|2x+5|^{2/3}$$

$$= \frac{2}{3}\log|2x+5|$$

$$= \frac{2}{3}\frac{\ln|2x+5|}{\ln 10}$$

* Differentiating yields

$$f'(x) = \frac{2}{3}\cdot\frac{1}{\ln 10}\cdot\frac{1}{2x+5}(2)$$

$$= \boxed{\frac{4}{3(2x+5)\ln 10}}.$$

Example (5) : Page (413)

If $f(x) = x^x$ and $x>0$, find $D_x y$.

Solution

$$y = x^x$$

* We use the method of logarithmic differentiation .

* In this case we take the natural logarithm of both sides of the equation $y = x^x$.

$$\ln y = \ln x^x = x\ln x$$

* we differentiate implicitly as follows :

$$D_x(\ln y) = D_x(x\ln x)$$

$$\frac{1}{y} D_x y = 1 + \ln x$$

$$D_x y = y\,(1 + \ln x)$$

$$= \boxed{x^x\,(1 + \ln x)}.$$

Exercise 7.5:No. **1, 3, 5, 7, 11, 15, 19, 21, 27(b), 29(b), 31.**

CHAPTER (8)

INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

8.1 INVERSE TRIGONOMETRICFUNCTIONS : Page (426)

Definition (8.1) : Page (426)

The inverse sine function, denoted $\sin^{-1}$, is defined by

$$y = \sin^{-1} x \quad \text{if and only if} \quad x = \sin y$$

For $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$.

Figure 8.1

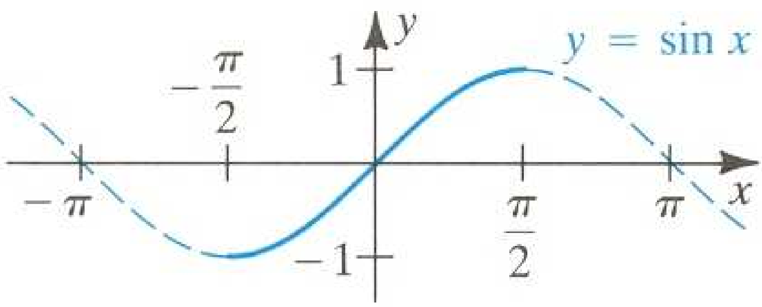


Figure 8.2

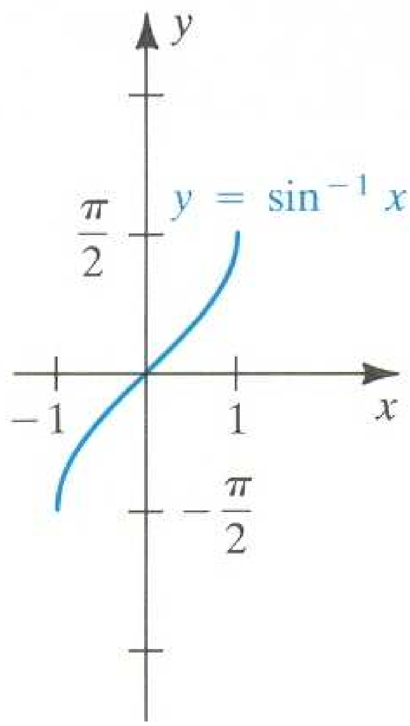


ILLUSTRATION : Page (426)

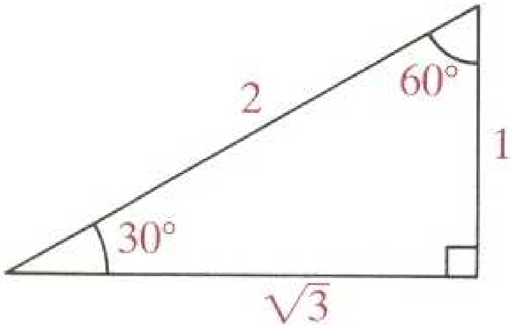
* If $y = \sin^{-1}\frac{1}{2}$, then $\sin y = \frac{1}{2}$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

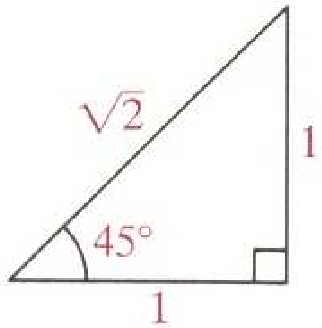
Hence $\boxed{y = \frac{\pi}{6}}$.

* If $y = \arcsin\left(-\frac{1}{2}\right)$, then $\sin y = -\frac{1}{2}$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Hence $\boxed{y = -\dfrac{\pi}{6}}$.

Remember that :





Properties of $\sin^{-1}$ (8.2) : Page (427)

(i) $\sin\left(\sin^{-1} x\right) = \sin\left(\arcsin x\right) = x \quad \text{if} \quad -1 \le x \le 1$.

(ii) $\sin^{-1}\left(\sin x\right) = \arcsin\left(\sin x\right) = x \quad \text{if} \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$.

Remember that :

* $f\left(f^{-1}(x)\right) = x \quad , \quad f^{-1}\left(f(x)\right) = x$

ILLUSTRATION : Page (427)

* $\sin\left(\sin^{-1}\dfrac{1}{2}\right) = \boxed{\dfrac{1}{2}}$ since $-1 < \dfrac{1}{2} < 1$.

* $\arcsin\left(\sin\dfrac{\pi}{4}\right) = \boxed{\dfrac{\pi}{4}}$ since $-\dfrac{\pi}{2} < \dfrac{\pi}{4} < \dfrac{\pi}{2}$.

* $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right) = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \boxed{\dfrac{\pi}{3}}$.

Definition (8.3) : *Page (427)*

The inverse cosine function, denoted $\cos^{-1}$ *, is defined by*

$$y = \cos^{-1} x \quad \text{if and only if} \quad x = \cos y$$

For $-1 \le x \le 1$ *and* $0 \le y \le \pi / 2$.

Figure 8.3

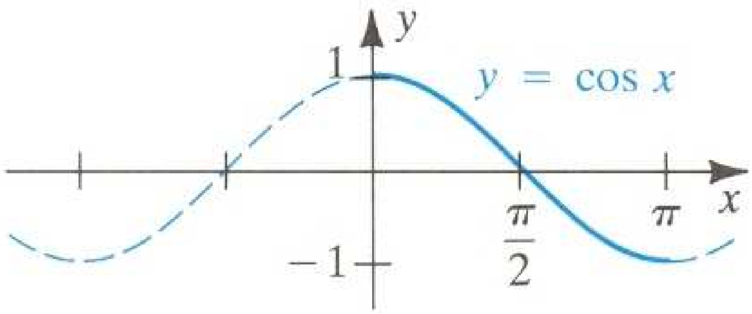


Figure 8.4

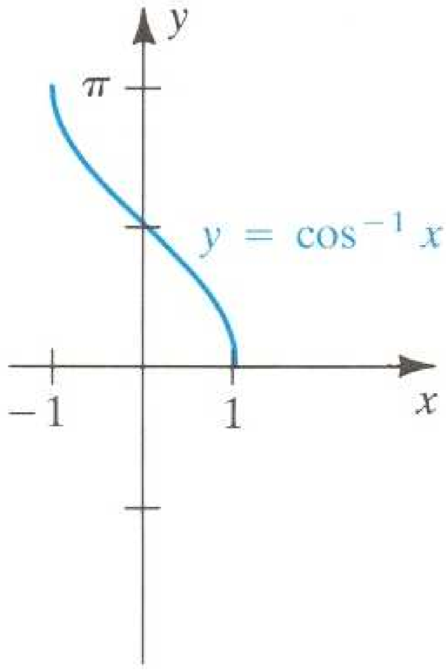


ILLUSTRATION : *Page (427)*

* *If* $y = \cos^{-1} \dfrac{1}{2}$, *then* $\cos y = \dfrac{1}{2}$ *and* $0 \le y \le \pi$.

Hence $\boxed{y = \dfrac{\pi}{3}}$.

* *If* $y = \arccos\left(-\dfrac{1}{2}\right)$, *then* $\cos y = -\dfrac{1}{2}$ *and* $0 \le y \le \pi$.

Hence $\boxed{y = \dfrac{2\pi}{3}}$.

Properties of $\cos^{-1}$ (8.4) : Page (428)

(i) $\cos\left(\cos^{-1} x\right) = \cos\left(\arccos x\right) = x \quad \text{if} \quad -1 \le x \le 1$.

(ii) $\cos^{-1}\left(\cos x\right) = \arccos\left(\cos x\right) = x \quad \text{if} \quad 0 \le x \le \pi$.

ILLUSTRATION : Page (428)

* $\cos\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = \boxed{-\frac{1}{2}}$ since $-1 < -\frac{1}{2} < 1$.

* $\arccos\left(\cos\frac{2\pi}{3}\right) = \boxed{\frac{2\pi}{3}}$ since $0 < \frac{2\pi}{3} < \pi$.

* $\cos^{-1}\left[\cos\left(-\frac{\pi}{4}\right)\right] = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{\pi}{4}}$.

Definition (8.5) : Page (428)

The inverse tangent function , or arctangent function denoted $\tan^{-1}$, or $\arctan$ is defined by

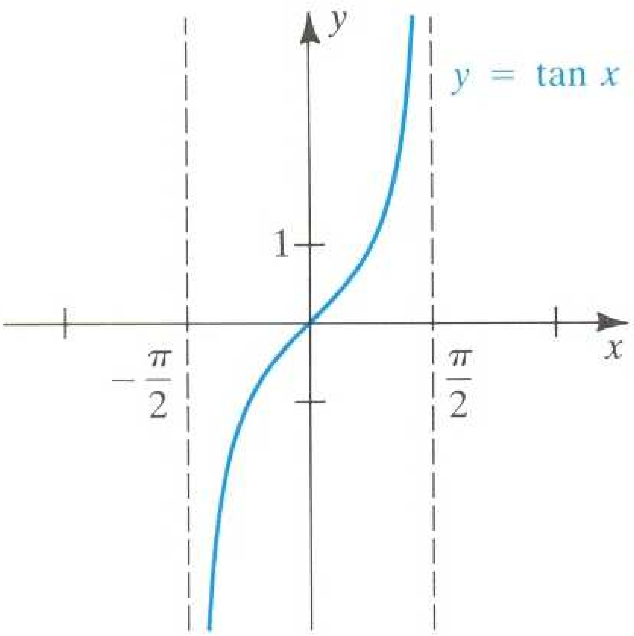
$$y = \tan^{-1} x \quad \text{if and only if} \quad x = \tan y$$

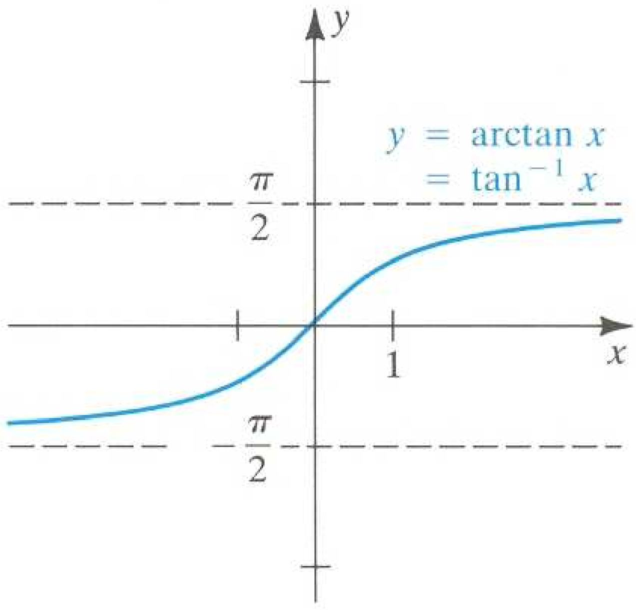
For every x and $-\pi/2 \le y \le \pi/2$.

Figure 8.5

Figure 8.6

$y = \tan x,\ -\pi/2 \le x \le \pi/2$





Properties of $\tan^{-1}$ (8.6) : *Page (429)*

(i) $\tan\left(\tan^{-1} x\right) = \tan\left(\arctan x\right) = x$ for every x .

(ii) $\tan^{-1}\left(\tan x\right) = \arctan\left(\tan x\right) = x$ if $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

ILLUSTRATION : *Page (428)*

* If $y = \arctan\left(-1\right)$, then $\tan y = -1$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

  Hence $\boxed{y = -\dfrac{\pi}{4}}$.

* $\tan\left(\tan^{-1} 1000\right) = \boxed{1000}$ by (8.6) (i) .

* $\tan^{-1}\left(\tan\dfrac{\pi}{4}\right) = \boxed{\dfrac{\pi}{4}}$ since $-\dfrac{\pi}{2} < \dfrac{\pi}{4} < \dfrac{\pi}{2}$.

* $\arctan\left(\tan 0\right) = \arctan 0 = \boxed{0}$.

Definition (8.7) : *Page (430)*

The inverse secat function , or arcsecant function denoted $\sec^{-1}$ *, or* $\text{arc}\sec$ *is defined by*

$$y = \sec^{-1} x \quad \text{if and only if} \quad x = \sec y$$

For $|x| \geq 1$ *and* y *in* $[0, \pi/2)$ *or in* $[\pi, 3\pi/2)$ *.*

Figure 8.8

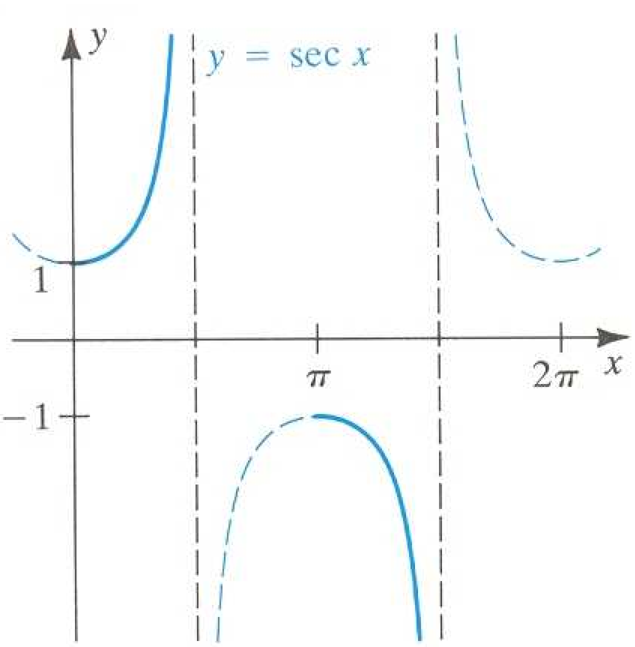
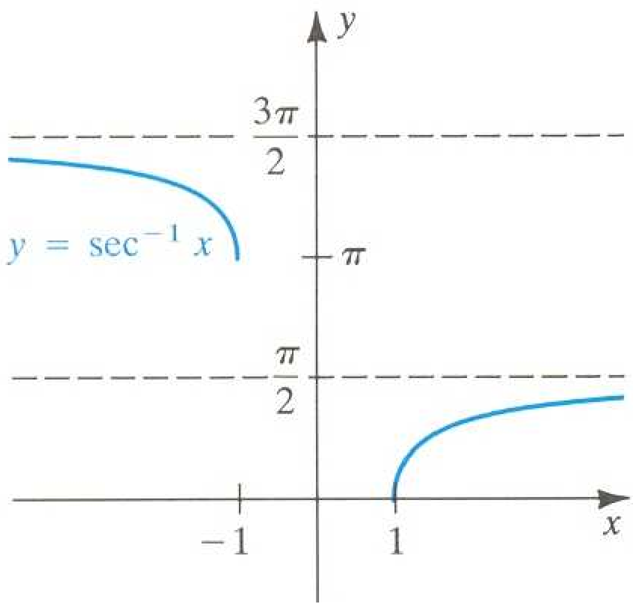


Figure 8.9



Example (3) : *Page (431)*

If $-1 \leq x \leq 1$ *, rewrite* $\cos\left(\sin^{-1} x\right)$ *as an algebraic expression in* x *.*

Solution

** Let*

$$y = \sin^{-1} x \;, \quad \text{or , equivalently ,} \quad \sin y = x \;.$$

** We wish to express* $\cos y$ *in terms of* x *. Since* $-\pi/2 \leq y \leq \pi/2$ *, it follows that* $\cos y \geq 0$ *, and hence*

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \;.$$

** Consequently* $\quad \cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2} \;.$

* *The last identity can also be geometrically if* $0 < x < 1$. *In this case* $0 < y < \pi / 2$ *, and we may regard* y *as the radian measure of an angle of a right triangle such that* $\sin y = x$ *, as illustrated in Figure 8.11 . (The side of length is found by using the Pythagorean theorem) .*

Remember that :

* ***Pythagorean theorem***

$$c^2 = a^2 + b^2 \Rightarrow c = \sqrt{a^2 + b^2}$$

$$a^2 = c^2 - b^2 \Rightarrow a = \sqrt{c^2 - b^2}$$

$$b^2 = c^2 - a^2 \Rightarrow b = \sqrt{c^2 - a^2}$$

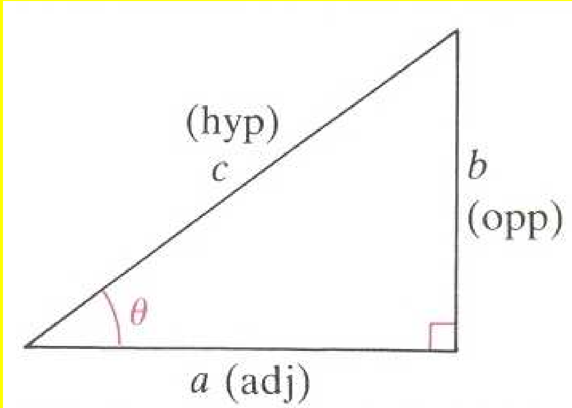
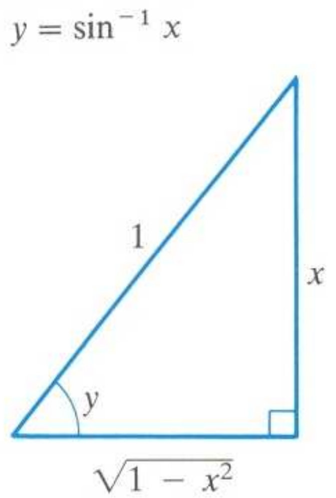


Figure 8.11



* ***Referring to the triangle , we have***

$$\cos\left(\sin^{-1} x\right) = \cos y = \frac{\sqrt{1 - x^2}}{1} = \boxed{\sqrt{1 - x^2}}$$

Exercise 8.1: No: **1, 9.**

8.2 DERIVATIVES AND INTEGRALS : ***Page (426)***

Theorem (8.8) : ***Page (434)***

$$(i)\ D_x \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} D_x u .$$

$$(ii)\ D_x \cos^{-1} u = -\frac{1}{\sqrt{1-u^2}} D_x u .$$

$$(iii)\ D_x \tan^{-1} u = \frac{1}{1+u^2} D_x u .$$

$$(iv)\ D_x \sec^{-1} u = \frac{1}{u\sqrt{u^2-1}} D_x u .$$

ILLUSTRATION : *Page (435)*

$f(x)$	$f'(x)$
* $\sin^{-1} 3x$	$\frac{1}{\sqrt{1-(3x)^2}} D_x(3x) = \boxed{\frac{3}{\sqrt{1-9x^2}}}$.
* $\arccos(\ln x)$	$-\frac{1}{\sqrt{1-(\ln x)^2}} D_x \ln x = \boxed{-\frac{1}{x\sqrt{1-(\ln x)^2}}}$.
* $\tan^{-1} e^{2x}$	$\frac{1}{1+\left(e^{2x}\right)^2} D_x e^{2x} = \boxed{\frac{2e^{2x}}{1+e^{4x}}}$.
* $\operatorname{arc\,sec}\left(x^2\right)$	$\frac{1}{x^2\sqrt{\left(x^2\right)^2-1}} D_x\left(x^2\right) = \boxed{\frac{2}{x\sqrt{x^4-1}}}$.

Theorem (8.9) : *Page (434)*

For $a > 0$,

$$(i)\ \int \frac{1}{\sqrt{a^2-u^2}}\,du=\sin^{-1}\frac{u}{a}+C\ .$$

$$(ii)\ \int \frac{1}{a^2+u^2}\,du=\frac{1}{a}\tan^{-1}\frac{u}{a}+C\ .$$

$$(iii)\ \int \frac{1}{u\sqrt{u^2-a^2}}\,du=\frac{1}{a}\sec^{-1}\frac{u}{a}+C\ .$$

Example (2) : *Page (437)*

Evaluate $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$.

Solution

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx=\int \frac{e^{2x}}{\sqrt{1-\left(e^{2x}\right)^2}}\,dx$$

* *The integral may be written as in Theorem (8.9) (i) by letting* $a=1$ *and using the substitution*

$$\boxed{u=e^{2x}}$$

$$du=2e^{2x}\,dx \quad\Rightarrow\quad \frac{1}{2}du=e^{2x}\,dx$$

* *We proceed as follows :*

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$$

Remember that :

$$*\ \int \frac{1}{\sqrt{a^2-u^2}}\,du=\sin^{-1}\frac{u}{a}+C$$

$$= \frac{1}{2}\sin^{-1} u + C$$

$$= \boxed{\frac{1}{2}\sin^{-1} e^{2x} + C}\ .$$

Example (3) : *Page (438)*

Evaluate $\int \frac{x^2}{5+x^6}\,dx$.

Solution

$$\int \frac{x^2}{5+x^6}\,dx = \int \frac{x^2}{5+\left(x^3\right)^2}\,dx$$

* *The integral may be written as in Theorem (8.9) (ii) by letting* $a^2 = 5$ *and using the substitution*

$$\boxed{u = x^3}$$

$$du = 3x^2\,dx \quad \Rightarrow \quad \frac{1}{3}du = x^2\,dx$$

* *We proceed as follows :*

$$\int \frac{x^2}{5+x^6}\,dx = \frac{1}{3}\int \frac{1}{\left(\sqrt{5}\right)^2 + u^2}\,du$$

Remember that :

$$* \quad \int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$= \frac{1}{3}\cdot\frac{1}{\sqrt{5}}\tan^{-1}\frac{u}{\sqrt{5}} + C$$

$$= \boxed{\frac{\sqrt{5}}{15}\tan^{-1}\frac{x^3}{\sqrt{5}}+C}.$$

Example (4) : *Page (438)*

Evaluate $\displaystyle\int \frac{1}{x\sqrt{x^4-9}}\,dx$.

Solution

$$\int \frac{1}{x\sqrt{x^4-9}}\,dx = \int \frac{1}{x\sqrt{\left(x^2\right)^2-9}}\,dx$$

* *The integral may be written as in Theorem (8.9) (iii) by letting* $a^2=9$ *and using the substitution*

$$\boxed{u=x^2}$$

$$du = 2x\,dx$$

$$\Rightarrow \quad \frac{1}{2}du = x^2\frac{dx}{x} \quad \Rightarrow \quad \frac{1}{2}\frac{du}{u}=\frac{dx}{x}$$

* *We proceed as follows :*

$$\int \frac{1}{x\sqrt{x^4-9}}\,dx = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-3^2}}\,du$$

Remember that :

$$* \quad \int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\frac{u}{a}+C$$

$$= \frac{1}{2}\cdot\frac{1}{3}\sec^{-1}\frac{u}{3}+C$$

$$= \boxed{\frac{1}{6}\sec^{-1}\frac{x^2}{3}+C}.$$

Exercise 8.2: No: **1, 3, 14, 29 (a), 31 (a).**

8.3 HYPERBOLIC FUNCTIONS : ***Page (440)***

Definition (8.10) : ***Page (440)***

The hyperbolic sine function , denoted by sinh , and the hyperbolic cosine function , denoted by cosh , are defined by

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

for every real number x .

Figure 8.13

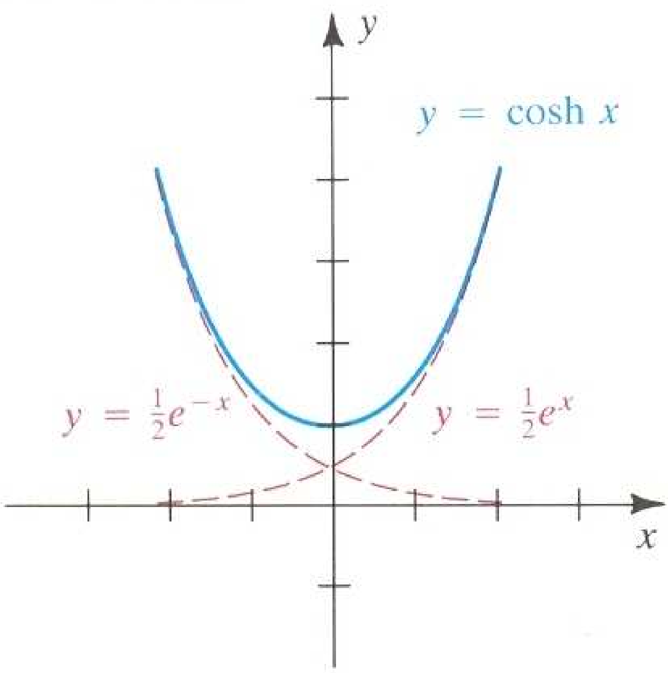
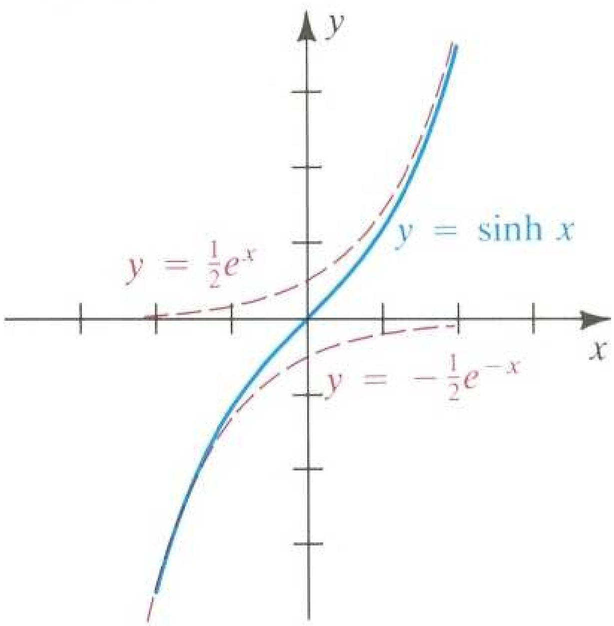


Figure 8.14



Theorem (8.11) : ***Page (441)***

$$\cosh^{2} x + \sinh^{2} x = 1 \ .$$

Theorem (8.12) : ***Page (442)***

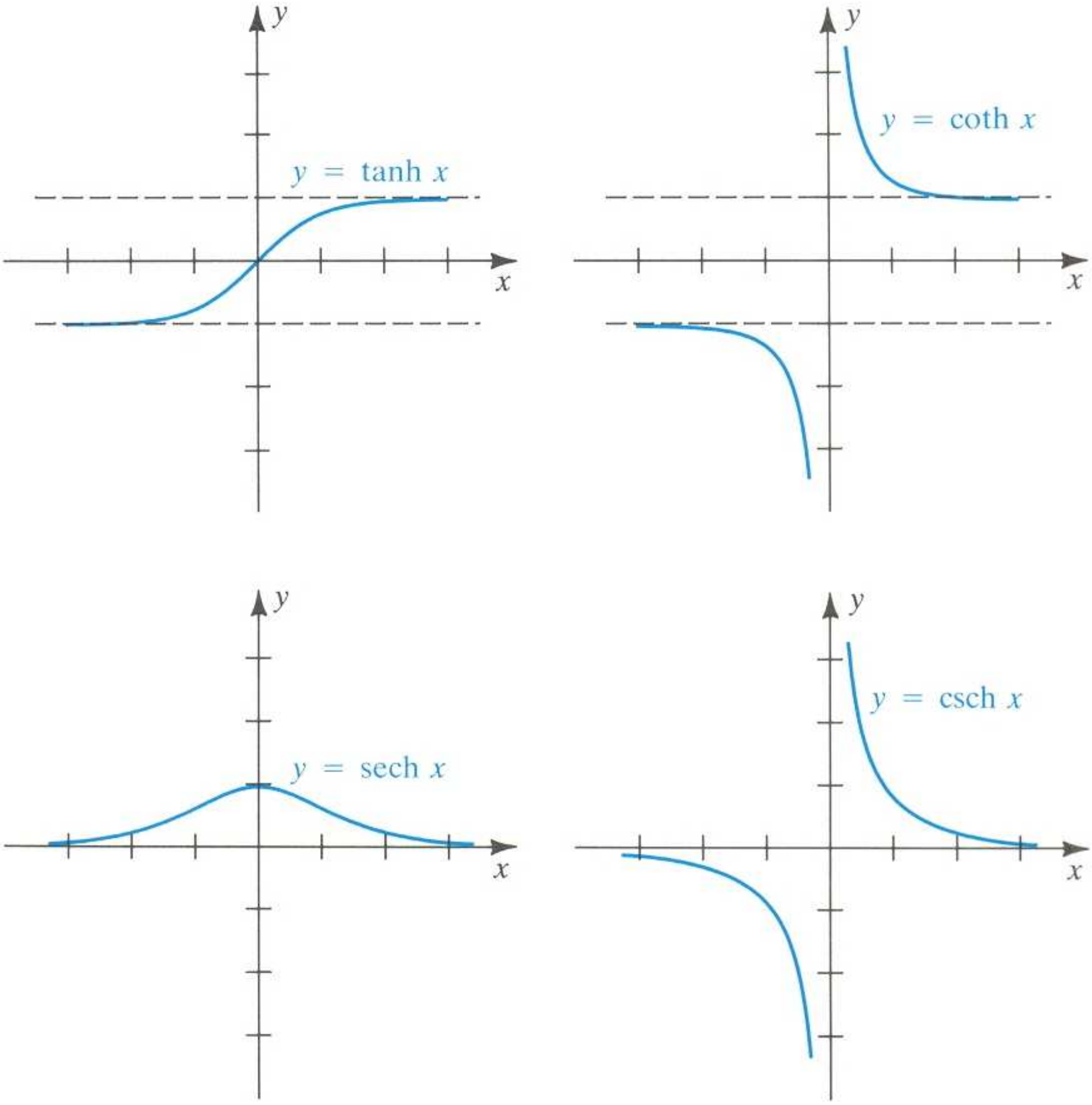
$$\text{(i)} \ \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \ .$$

(ii) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$, $x \neq 0$.

(iii) $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^{x} + e^{-x}}$.

(iv) $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^{x} - e^{-x}}$, $x \neq 0$.

Figure 8.18



Theorem (8.13) :   ***Page (443)***

$$\text{(i) } 1-\tanh^2 x=\operatorname{sech}^2 x . \qquad \text{(ii) } \coth^2 x-1=\operatorname{csch}^2 x .$$

Theorem (8.14) : ***Page (444)***

$$\text{(i) } D_x \sinh u=\cosh u \; D_x u .$$
$$\text{(ii) } D_x \cosh u=\sinh u \; D_x u .$$
$$\text{(iii) } D_x \tanh u=\operatorname{sech}^2 u \; D_x u .$$
$$\text{(iv) } D_x \coth u=-\operatorname{csch}^2 u \; D_x u .$$
$$\text{(v) } D_x \operatorname{sech} u=-\operatorname{sech} u \tanh u \; D_x u .$$
$$\text{(vi) } D_x \operatorname{csch} u=-\operatorname{csch} u \coth u \; D_x u .$$

Example (1) : ***Page (444)***

If $f(x)=\cosh\left(x^2+1\right)$ *, find* $f'(x)$.

Solution

$$f(x)=\cosh\left(x^2+1\right)$$

** Applying Theorem (8.14) (i) , with* $u=x^2+1$ *, we obtain*

Remember that :

$$* \; \boxed{D_x \sinh u=\cosh u \, D_x u}$$

$$f'(x)=\sinh\left(x^2+1\right).D_x\left(x^2+1\right)$$

$$=\boxed{2x\sinh\left(x^2+1\right)} .$$

Theorem (8.15) : ***Page (445)***

$$\text{(i) } \int \sinh u \, du=\cosh u+C .$$

(ii) $\int \cosh u \, du = \sinh u + C$.

(iii) $\int \operatorname{sech}^2 u \, du = \tanh u + C$.

(iv) $\int \operatorname{csch}^2 u \, du = -\coth u + C$.

(v) $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$.

(vi) $\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$.

Example (3) : *Page (445)*

Evaluate $\int x^2 \sinh x^3 \, dx$.

Solution

$$\int x^2 \sinh x^3 \, dx$$

* The integral may be written as in Theorem (8.15) (i) by using the substitution

$$\boxed{u = x^3}$$

$$du = 3x^2 \, dx \quad \Rightarrow \quad \frac{1}{3} du = x^2 \, dx$$

* We proceed as follows :

$$\int x^2 \sinh x^3 \, dx = \frac{1}{3} \int \sinh u \, du$$

Remember that :

* $\int \sinh u \, du = \cosh u + C$

$$= \frac{1}{3} \cosh u + C$$

$$= \boxed{\frac{1}{3} \cosh x^3 + C} .$$

Exercise 8.3: No: 3, 5, 7, 13, 14, 17, 21, 27, 41.

8.4 INVERSE HYPERBOLIC FUNCTIONS : Page (447)

Theorem (8.16) : Page (448)

(i) $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

(ii) $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$, $x \geq 1$.

(iii) $\tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}$, $|x| < 1$.

(iv) $\operatorname{sech}^{-1} x = \ln \frac{1 + \sqrt{1 - x^2}}{x}$, $0 < x < 1$.

Theorem (8.17) : Page (449)

(i) $D_x \sinh^{-1} u = \frac{1}{\sqrt{u^2 + 1}} D_x u$.

(ii) $D_x \cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}} D_x u$, $u > 1$.

(iii) $D_x \tanh^{-1} u = \frac{1}{1 - u^2} D_x u$, $|u| < 1$.

(iv) $D_x \operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1 - u^2}} D_x u$, $0 < u < 1$.

Example (1) : Page (449)

If $y = \sinh^{-1}(\tan x)$, find dy / dx.

Solution

$$y = \sinh^{-1}(\tan x)$$

* Using Theorem (8.14) (i) with $u = \tan x$, we have

Remember that :

$$* \quad \boxed{D_x \sinh^{-1} u = \frac{1}{\sqrt{u^2+1}} D_x u}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\tan^2 x + 1}} \frac{d}{dx} \tan x$$

$$= \frac{1}{\sqrt{\sec^2 x}} \sec^2 x$$

$$= \frac{1}{|\sec x|} |\sec x|^2 = \boxed{|\sec x|} .$$

Theorem (8.18) : Page (450)

$$\text{(i)} \int \frac{1}{\sqrt{a^2+u^2}} du = \sinh^{-1} \frac{u}{a} + C \quad , \quad a > 0 .$$

$$\text{(ii)} \int \frac{1}{\sqrt{u^2-a^2}} du = \cosh^{-1} \frac{u}{a} + C \quad , \quad 0 < a < u .$$

$$\text{(iii)} \int \frac{1}{a^2-u^2} du = \frac{1}{a} \tanh^{-1} \frac{u}{a} + C \quad , \quad |u| < a .$$

$$\text{(iv)} \int \frac{1}{u\sqrt{a^2-u^2}} du = -\frac{1}{a} \operatorname{sech}^{-1} \frac{|u|}{a} + C \quad , \quad 0 < |u| < a .$$

Example (2) : Page (438)

Evaluate $\displaystyle\int \frac{1}{\sqrt{25+9x^2}} dx$.

Solution

$$\int \frac{1}{\sqrt{25+9x^2}}\,dx = \int \frac{1}{\sqrt{25+(3x)^2}}\,dx$$

* *The integral may be written as in Theorem (8.18) (i) by letting* $a^2 = 25$ *and using the substitution*

$$\boxed{u = 3x}$$

$$du = 2\ dx \quad \Rightarrow \quad \frac{1}{3}du = dx$$

* *We proceed as follows :*

$$\int \frac{1}{\sqrt{25+9x^2}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{5^2+u^2}}\,du$$

Remember that :

$$* \quad \int \frac{1}{\sqrt{a^2+u^2}}\,du = \sinh^{-1}\frac{u}{a} + C \ , \quad a > 0$$

$$= \frac{1}{3}\sinh^{-1}\frac{u}{5} + C$$

$$= \boxed{\frac{1}{3}\sinh^{-1}\frac{3x}{5} + C} .$$

Example (3) : *Page (451)*

Evaluate $\displaystyle\int \frac{e^x}{16-e^{2x}}\,dx$.

Solution

$$\int \frac{e^x}{16-e^{2x}}\,dx = \int \frac{e^x}{16-(e^x)^2}\,dx$$

* *The integral may be written as in Theorem (8.18) (iii) by letting* $a^2 = 16$ *and using the substitution*

$$\boxed{u = e^x}$$

$$du = e^x \, dx$$

* *We proceed as follows :*

$$\int \frac{e^x}{16 - e^{2x}} \, dx = \int \frac{1}{4^2 - u^2} \, du$$

Remember that :

$$* \quad \int \frac{1}{a^2 - u^2} \, du = \frac{1}{a} \tanh^{-1} \frac{u}{a} + C \quad , \quad |u| > a$$

$$= \frac{1}{4} \tanh^{-1} \frac{u}{4} + C$$

$$= \boxed{\frac{1}{4} \tanh^{-1} \frac{e^x}{4} + C}$$

for $|u| < a$ *(that is ,* $e^x < 4$ *) .*

Exercise 8.4: No: 3, 5, 7, 13, 14.

TAIBAH UNIVERSITY

FACULTY OF SCIENCE

Department of Mathematics

SYLLABUS

Course # : Math 101

Course Title : Calculus (1)

Textbook : Calculus by Earl W. Swokowski, Fifth Edition.

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