محاضرة في رياضيات

د. مجدي

③ دالة لاجندر page "147"

المعادلة التفاضلية

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$y_1(x) = [\text{even Terms}]a_0$

$y_2(x) = [\text{odd Terms}]a_1$

$y(x) = [x^n - \cancel{c}x^{n-1} + x^{n-2} - \dots]a_n$

$$a_n = \frac{(2n)!}{2^n (n!)^2}$$

مش مطالب بيهم تعرفهم "للتوضيح فقط"

$$P_n(x) = \sum_{s=0}^{\infty} \frac{(-)^s (2n-2s)!\, x^{n-2s}}{2^n\, s!\,(n-s)!\,(n-2s)!} = P_n(x).$$

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2-1)^n$$

لاثبات المعادلة التفاضلية

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$U = (x^2-1)^n \qquad D^n = \frac{d^n}{dx^n}$

$DU = D(x^2-1)^n = n(x^2-1)^{n-1}\, 2x$

$DU = 2nx(x^2-1)^{n-1}$

$$DU = 2nx\frac{(x^2-1)^n}{(x^2-1)} \Rightarrow (x^2-1)DU = 2nx(x^2-1)^n$$

$$D^{n+1}[(x^2-1)DU] = 2n\, D^{n+1}[x(x^2-1)^n]$$

$$\frac{d^n}{dx^n}(U.V) = U.V^{(n)} + \frac{n}{1!}U^{(1)}V^{(n-1)} + \frac{n(n-1)}{2!}U^{(2)}.V^{(n-2)} + \dots + U^{(n)}.V$$

$D^{n+1}[(x^2-1)DU] = 2n\,D^{n+1}[x(x^2-1)^n] \leftarrow$ بتطبيق الحل على المعادلة

$U = (x^2-1) \qquad V = DU$

$$(x^2-1)D^{n+2}U + \frac{(n+1)}{1!}(2x)D^{n+1}U + \frac{n(n+1)}{2!}2.D^nU$$

$$= 2n \, \overset{n+1}{D}_x \left[x(x^2-1)^n \right] = 2n \left[x D^{n+1} (x^2-1)^n + (n+1) D^n (x^2-1)^n \right]$$

$$(x^2-1) D^{n+2} u + \cancel{2n x D^{n+1}} u + 2x D^{n+1} u + (n+1)n \, D^n u$$

$$= \cancel{2n x D^{n+1}} u + 2n(n+1) D^n u$$

$$(x^2-1) D^{n+2} u + 2x D^{n+1} u + n(n+1) D^n u - 2n(n+1) D^n u = 0$$

$$(x^2-1) D^{n+2} u + 2x D^{n+1} u - n(n+1) D^n u = 0$$

$$y = D^n u$$

$$\therefore (x^2-1) D^2 y + 2x D y - n(n+1) y = 0$$

$$\therefore (x^2-1) y'' + 2x y' - n(n+1) y = 0$$

$$D^n u = \frac{d}{dx^n} (x^2-1)^n \Longrightarrow$$ حل لمعادلة لاجندر التفاضلية

$\cancel{P_n(x) =}$

$$P_n(x) = A D^n (x^2-1)^n \Longrightarrow$$ حل لمعادلة لاجندر التفاضلية

* $P_0(x) = 1$ دالة كثيرة حدود من الدرجة الصفرية

$P_1(x) = x$ دالة كثيرة حدود من الدرجة الأولى

$P_2(x) = \frac{(3x^2-1)}{2}$ دالة كثيرة حدود من الدرجة الثانية

$P_3(x) = \frac{(5x^3-3x)}{2}$ دالة كثيرة حدود من الدرجة الثالثة

مهمة

$\boxed{1}$ $P'_{(n+1)}(x) = x P'_n(x) + (n+1) P_n(x)$

Proof

$$P_{n+1}(x) = \frac{1}{2^{n+1} (n+1)!} D^{n+1} (x^2-1)^{n+1} = \frac{1}{2^{n+1} (n+1)!} D^n D \left[(x^2-1)^{n+1} \right]$$

$$= \frac{1}{2^{\cancel{n+1}} \cancel{(n+1)!}} D^n \left[\cancel{(n+1)} (x^2-1)^n \cancel{2} x \right]$$

$n!$

$$P_{n+1}(x) = \frac{1}{2^n n!} D^n \left[x (x^2-1)^n \right]$$

$$= \frac{1}{2^n n!} \left[x \, D^n (x^2-1)^n + \frac{n}{1!} D^{n-1} (x^2-1)^n \right]$$

$$P_{n+1}(x) = x \, P_n(x) + \frac{n}{2^n n!} D^{n-1} (x^2-1)^n$$ هنحتاجها في الاثبات اللى جاي

$$P'_{n+1}(x) = x \, P'_n(x) + P_n(x) + \frac{n}{2^n n!} D^n (x^2-1)^n$$

$$= x P'_n(x) + P_n(x) + n \, P_n(x)$$

$$P'_{n+1}(x) = x P'_n(x) + (n+1) P_n(x) \quad \#$$

② $(n+1) P_{n+1}(x) = (2n+1) \, x \, P_n(x) - n \, P_{n-1}(x)$

proof.

$$(n+1)(x^2-1)^n = n(x^2-1)^n + (x^2-1)^n + n(x^2-1)^n - n(x^2-1)^n$$

$$= (x^2-1)^n + 2n \, (x^2-1)^{n-1}(x^2-1) - n(x^2-1)^n$$

$$= (x^2-1)^n + 2n x^2 (x^2-1)^{n-1} - 2n(x^2-1)^{n-1} - n(x^2-1)^n$$

بتفاضل الطرفين $(n-1)$ من المرات

$$(n+1) D^{n-1}(x^2-1)^n = D^{n-1}(x^2-1)^n + 2n \, D^{n-1}\left[x^2 (x^2-1)^{n-1}\right]$$

$$- 2n D^{n-1}(x^2-1)^{n-1} - n D^{n-1}(x^2-1)^n$$

$$= D^n \left[x (x^2-1)^n \right] - 2n D^{n-1} (x^2-1)^{n-1}$$

$$- n D^{n-1} (x^2-1)^n$$

بالضرب في $\frac{1}{2^n \; n!}$

$$\frac{n+1}{2^n n!} \cancel{(x^2-1)^n} D^{n-1}(x^2-1)^n = \frac{1}{2^n n!} D^n \left[x (x^2-1)^n \right]$$

$$- \frac{2n}{2^n n!} D^{n-1}(x^2-1)^{n-1} - \frac{n}{2^n n!} D^{n-1}(x^2-1)^n$$

$$\frac{n}{2^n n!} D^{n-1} (x^2-1)^n + \frac{1}{2^n n!} D^{n-1} (x^2-1)^n$$

$$= \frac{1}{2^n n!} D^n \left[x(x^2-1)^n\right] - \frac{2n}{2^n n!} D^{n-1} (x^2-1)^{n-1}$$

$$- \frac{n}{2^n n!} D^{n-1} (x^2-1)^n$$

R.H.S

$$= \frac{1}{2^n n!} \left[x D^n (x^2-1)^n + n D^{n-1} (x^2-1)^n\right]$$

$$- \frac{1}{2^{n-1}(n-1)!} D^{n-1} (x^2-1)^{n-1} - \frac{n}{2^n n!} D^{n-1} (x^2-1)^n$$

$$= \frac{x}{2^n n!} D^n (x^2-1)^n + \cancel{\frac{n}{2^n n!} D^{n-1} (x^2-1)^n}$$

$$- \frac{1}{2^{n-1}(n-1)!} D^{n-1} (x^2-1)^{n-1} - \cancel{\frac{n}{2^n n!} D^{n-1} (x^2-1)^n}$$

$$= x P_n(x) - P_{n-1}(x) \qquad = \frac{(n+1)}{2^n n!} D^{n-1} (x^2-1)^n$$

$$\frac{n}{2^n n!} D^{n-1} (x^2-1)^n = x P_n(x) - P_{n-1}(x) - \frac{1}{2^n n!} D^{n-1} (x^2-1)^n$$

في اثبات العلاقة الأولى :-

$$P_{n+1}(x) = x P_n(x) + \frac{n}{2^n n!} D^{n-1} (x^2-1)^n$$

$$\frac{n}{2^n n!} D^{n-1} (x^2-1)^n = P_{n+1}(x) - x P_n(x)$$

باقي الاثبات في آخر صفحة

③ $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$

Proof

From eq ②

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

فاضل الطرفين بالنسبة لـ X

$$(n+1)P'_{n+1}(x) = (2n+1)[xP'_n(x) + P_n(x)] - nP'_{n-1}(x)$$

From eq ① $P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x)$

بالطرح

$$nP'_{n+1}(x) = 2nxP'_n(x) + [(2n+1)-(n+1)]P_n(x) - nP'_{n-1}(x)$$

$$\cancel{n}P'_{n+1}(x) = 2\cancel{n}x\cdot P'_n(x) + \cancel{n}P_n(x) - \cancel{n}P'_{n-1}(x)$$

$$P'_{n+1}(x) = 2xP'_n(x) + P_n(x) - P'_{n-1}(x)$$

From eq ①

$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x)$$

$$[xP'_n(x) + (n+1)P_n(x)] = 2xP'_n(x) + P_n(x) - P'_{n-1}(x)$$

$$nP_n(x) + \cancel{P_n(x)} = \cancel{2}xP'_n(x) - \cancel{xP'_n(x)} + \cancel{P_n(x)} - P'_{n-1}(x)$$

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$

④ $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

Page :
Date :

Proof

$\boxed{R_1}$ $P'_{n+1}(X) = XP'_n(X) + (n+1)P_n(X)$

$\boxed{R_3}$ $nP_n(X) = XP'_n(X) - P'_{n-1}(X)$

بطرح R_3 من R_1

$P'_{n+1}(X) - nP_n(X) = (n+1)P_n(X) + P'_{n-1}(X)$

$P'_{n+1}(X) - P'_{n-1}(X) = (2n+1)P_n(X)$ ※

⑤ $(X^2-1)P'_n(X) = n\left[XP_n(X) - P_{n-1}(X)\right]$

Proof

استبدل n بـ $(n-1)$ في R_1

$P'_n(X) = XP'_{n-1}(X) + nP_{n-1}(X)$ ①

اضرب R_3 في X

$nXP_n(X) = X^2P'_n(X) - XP'_{n-1}(X)$ ②

$X^2P'_n(X) = XP'_{n-1}(X) + nXP_n(X)$ ③

بطرح ① من ③

$(X^2-1)P'_n(X) = nXP_n(X) - nP_{n-1}(X)$

$(X^2-1)P'_n(X) = n\left[XP_n(X) - P_{n-1}(X)\right]$ ※

\# Generating Function of legendre Function.

$(1-2Xt+t^2)^{-1/2}$ is Generating Function of $P_n(X)$

$$(1-2Xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(X)$$

وتستخدم هذه العلاقة في حل الاثباتات الآتية:

K.M.S

Page :
Date :

Example 10 page 157

Prove That :-

(i) $P_n(1)=1$

Solution

at $x=1$

$\therefore (1-2t+t^2)^{-\frac{1}{2}} = ((1-t)^2)^{-\frac{1}{2}} = (1-t)^{-1}$

باستخدام نظرية ذو الحدين.

$= 1+t+t^2+t^3+\cdots = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} t^n P_n(x)$

$\therefore P_n(1)=1$

(ii) $P_n(-1)=(-1)^n$

Solution

at $x=-1$

$\sum_{n=0}^{\infty} t^n P_n(-1) = (1+2t+t^2)^{-\frac{1}{2}} = (1+t)^{-1}$

$= 1-t+t^2-t^3+\cdots = \sum_{n=0}^{\infty} (-1)^n t^n$

$\therefore P_n(-1) = (-1)^n$

~~Ex~~ (iii) $P_{2n+1}(0)=0$, $P_{2n}(0) = \dfrac{(-1)^n (2n)!}{2^{2n}(n!)^2}$

Solution

at $x=0$

$\therefore (1+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(0)$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad \cdots$

$P_0(0) + t\underline{P_1(0)} + t^2 P_2(0) + t^3 \underline{P_3(0)} \cdots = 1 - \frac{1}{2}t^2 + \frac{1.3}{2.2}t^4 - \frac{1.3.5}{2.2.2}t^6 + \cdots$

نستنتج من ذلك أن :

جميع الحدود الفردية = Zero

$P_{2n+1}(0) = 0$

$P_0(0) = 1 \quad , \quad P_2(0) = \frac{-1}{2} \quad , \quad P_4(0) = \frac{1\cdot 3}{2\cdot 2}$

$$\therefore P_{2n}(0) = \frac{(-)^n (2n)!}{2^{2n} (n!)^2}$$

ملحوظة

مفكوك $(Z+1)^n$ يساوي

$$(Z+1)^n = 1 + n + \frac{n(n-1)}{2!} Z^2 + \ldots\ldots$$

Definite integrals involving legendre polynomia

$$I_n = \int_{-1}^{1} F(X)\, P_n(X)\, dX$$

$$= \frac{1}{2^n n!} \int_{-1}^{1} F(X)\, D^n (X^2-1)^n\, dX$$

~~let $D^n(X^2-1)^n dX = dv$~~

~~$\therefore dv = D\, D^{n-1} (X^2-1)^n dX$~~

~~$dv = \frac{d}{dX}\left[D^{n-1}(X^2-1)^n\right] dX$~~

~~$I_n = \frac{1}{2^n n!}\left[F(X)\, D^{n-1}(X^2-1)^n\right]$~~

let $dv = D^n (X^2-1)^2\, dX \qquad u = F(X)$

$dv = D\, D^{n-1} (X^2-1)^n dX \qquad du = F'(X)$

$$dv = \frac{d}{dX}\left[D^{n-1}(X^2-1)^n\right] dX$$

$$v = D^{n-1}(X^2-1)^n$$

Page :
Date :

$$\therefore I_n = \frac{1}{2^n n!}\left[F(x)\, D^{n-1}(x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1} F'(x)\, D^{n-1}(x^2-1)^n dx$$

$$= \frac{-1}{2^n n!}\int_{-1}^{1} F'(x)\, D^{n-1}(x^2-1)^n dx$$

$$I_n = \frac{(-1)^n}{2^n n!}\int_{-1}^{1}(x-1)^n F^{(n)}(x)\, dx.$$

$$\int_{-1}^{1} F(x) P_n(x) dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1}(x^2-1) F^{(n)}(x)\, dx$$

if $F(x) = P_m(x)$; $m \neq n$ $\quad \therefore \int_{-1}^{1} P_m(x) P_n(x) dx = 0$

if $F(x) = P_n(x)$ $\quad \therefore \int_{-1}^{1}[P_n(x)]^2 dx = \frac{2}{2n+1}$

※ $F(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x) \cdots$

$$= \sum_{n=0}^{\infty} a_n P_n(x).$$

$$\int_{-1}^{1} F(x)\, P_n(x)\, dx = \int_{-1}^{1} a_n [P_n(x)]^2 dx \;;\quad n = 0, 1, 2 \cdots$$

$$= a_n \cdot \frac{2}{2n+1} \Rightarrow a_n = \frac{2n+1}{2}\int_{-1}^{1} F(x) P_n(x) dx$$

Ex;

$$F(x) = 3x^2 + 2x + 4 = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$$

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} F(x) P_n(x) dx$$

$$a_0 = \frac{1}{2}\int_{-1}^{1}(3x^2+2x+4) \cdot P_0(x)\, dx = 5$$

at $n=1$

$$a_1 = \frac{3}{2} \int_{-1}^{1} (3x^2+2x+4)\, P_1(x)\, dx = 2$$

$$a_2 = \frac{5}{2} \int_{-1}^{1} (3x^2+2x+4)\, P_2(x)\, dx = 2$$

$$\therefore F(x) = 5P_0(x) + 2P_1(x) + 2P_2(x) + \cdots\cdots$$

مطلوب تسليم Report أخر معاد بيوم الأربعاء القادم الموافق 2017/4/2

ال Report أخر صفحة بالكتاب ماعدا السؤال رقم 6 وبدلها كتابة اثباتات دالة ليسيل ولاجندر كاملة ☺

"لا تبخل علينا بالدعاء" ☺

$$P_{n+1}(x) - xP_n(x) = xP_n(x) - P_{n-1}(x) - \frac{1}{2^n n!} D^{n-1}(x^2-1)^n$$

$$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x) - \frac{1}{2^n n!} D^{n-1}(x^2-1)^n$$

$$nP_{n+1}(x) = 2nxP_n(x) - nP_{n-1}(x) - \left[P_{n+1}(x) - xP_n(x)\right]$$

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad \#$$